相対論における
「ねじれ」を伴う現象と接触幾何

大阪公立大学 南部陽一郎物理学研究所
石原 秀樹

目次

1. 導入
2. 接触空間に関する言葉使い
3. Null対称性をもった南部 - 後藤ストリングの古典解
4. 接触計量空間を用いたEinstein方程式の解
   - 3次元佐々木空間を用いた静的非一様時空
   - 時間的な3次元擬佐々木空間を用いたGödel宇宙
   - $T^{p,q}$ 空間を用いた高次元時空
5. 電磁波の偏光面の回転
   - Beltrami磁場中のFaraday効果

1. 導入

電場と磁場

$$\boldsymbol{E} = -\mathrm{grad}\ \phi \qquad \boldsymbol{F} = q\boldsymbol{E}$$

$$\boldsymbol{B} = \mathrm{rot}\,\boldsymbol{A} \qquad \boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B}$$

$$E_a := F_{ab}(\partial_t)^b$$

Polar vector

$$B_a := ({}^*F)_{ab}(\partial_t)^b = (\frac{1}{2}\epsilon_{abcd}F)^{cd}(\partial_t)^b$$

Axial vector

Riemann曲率

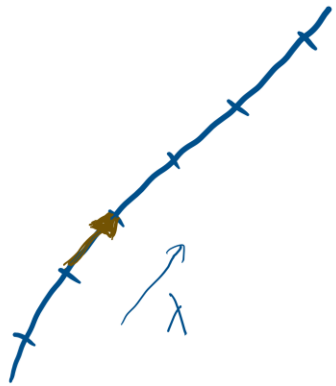
$$\text{electric part}: \quad E_{ac} := R_{abcd}(\partial_t)^b(\partial_t)^d$$

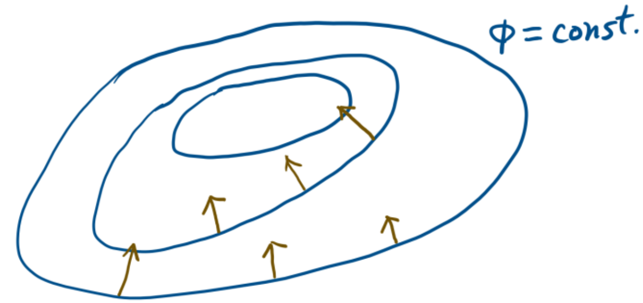
$$\text{magnetic part}: \quad B_{ac} :={}^*R_{abcd}(\partial_t)^b(\partial_t)^d$$

$$:= \frac{1}{2}\epsilon_{abef}R^{ef}{}_{cd}(\partial_t)^b(\partial_t)^d \sim R_{ijk(t)}$$

$$\text{Schwarzschild}: \quad E_{ac} \neq 0,\ B_{ac} = 0$$ **Polar**

$$\text{Kerr}: \quad E_{ac} \neq 0,\ B_{ac} \neq 0$$ **Axial**



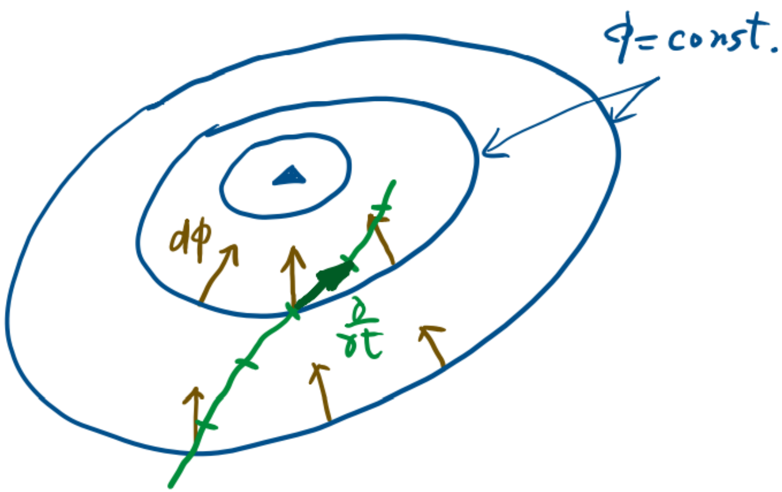


$$u = \frac{\partial}{\partial \lambda} = \partial_\lambda = \frac{\partial x^\mu}{\partial \lambda}\partial_\mu$$

tangent vector

$$\eta = d\phi = \frac{\partial \phi}{\partial x^\mu} dx^\mu$$

dual vector, co-vector, 1-form



$$\eta(u) \in \mathbb{R}$$

$$
\begin{aligned}
d\eta = d\eta_\mu \wedge dx^\mu &= \frac{\partial \eta_\mu}{\partial x^\nu} dx^\nu \wedge dx^\mu \\
&= \frac{\partial \eta_\mu}{\partial x^\nu} \frac{1}{2} (dx^\nu \wedge dx^\mu - dx^\mu \wedge dx^\nu) \\
&= \frac{1}{2} \left(\frac{\partial \eta_\mu}{\partial x^\nu} - \frac{\partial \eta_\nu}{\partial x^\mu} \right) dx^\nu \wedge dx^\mu
\end{aligned}
$$

3 dimension

$$
{}^* d\eta = \frac{1}{2} \epsilon_k{}^{ji} \left(\partial_j \eta_i - \partial_i \eta_j \right) dx^k \sim \mathrm{rot}\ \eta
$$

2. 接触空間に関する言葉使い

Contact manifold（接触多様体）

$\mathcal{M}$: $(2n+1)$ -dimensional manifold

$\eta_a = \eta_i \, (dx^i)_a$: 1-form on $\mathcal{M}$

Contact 1-form η_a on $\mathcal{M}$

$$\eta \wedge \underbrace{d\eta \wedge \cdots \wedge d\eta}_{n\text{ -times}} \neq 0$$

Reeb vector ξ^a

$$\eta_a \xi^a = \eta(\xi) = 1, \quad d\eta_{ab}\xi^a = d\eta(\xi, \quad) = \iota_\xi d\eta = 0$$

Simple example

$\mathbb{R}^{2n+1}$ の座標を $(x^1, y^1, x^2, y^2 \cdots, x^n, y^n, z)$ とする.

このとき，$\eta = dz + \sum_{i=1}^{n} x^i dy^i$ は，contact form である.

$$d\eta = \sum dx^i \wedge dy^i$$

$$\eta \wedge d\eta \wedge d\eta \cdots \wedge d\eta$$

$$= dz \wedge dx^1 \wedge dy^1 \wedge dx^2 \wedge dy^2 \cdots \wedge dx^n \wedge dy^n \neq 0$$

Reeb vector

$$\xi = \partial_z$$

$$\eta(\xi) = dz(\partial_z) = 1, \quad d\eta(\partial_z, \quad) = 0.$$

Contact metric manifold

Introduce a (1,1) tensor ϕ : (vector) $\to$ (vector) satifying

$$\boxed{\phi^2(X) = -X + \eta(X)\,\xi,} \quad \text{for arbitrary } X.$$

$$\phi^3(\xi) = \phi(\phi^2(\xi)) = \phi(-\xi + \eta(\xi)\xi) = \phi(-\xi + \xi) = 0$$

$$= \phi^2(\phi(\xi)) = -\phi(\xi) + \eta(\phi(\xi))\xi = -\phi(\xi) + \alpha\xi$$

$$\phi(\xi) = \alpha\xi$$

$$\alpha\phi(\xi) = \phi^2(\xi) = -\xi + \eta(\xi)\xi = -\xi + \xi = 0$$

$$\phi(\xi) = 0. \quad \text{: projection}$$

$$\phi^3(X) = \phi(\phi^2(X)) = \phi(-X + \eta(X)\xi) = \phi(-X) = -\phi(X)$$

$$= \phi^2(\phi(X)) = -\phi(X) + \eta(\phi(X))\xi$$

$$\eta(\phi(X)) = 0$$

Contact metric manifold

Introduce a (1,1) tensor ϕ : (vector) $\to$ (vector) satifying

$$\phi^2(X) = -X + \eta(X)\,\xi, \quad \text{for arbitrary } X.$$

If g and ϕ satisfy

$$g(\phi(X), \phi(Y)) = g(X, Y) - \eta(X)\eta(Y), \quad \text{for arbitrary } X, Y,$$

$(\mathcal{M}, \phi, \xi, \eta, g)$ is called an almost contact metric manifold.

$$g(\phi(\xi), \phi(\xi)) = g(\xi, \xi) - \eta(\xi)\eta(\xi) = 0$$

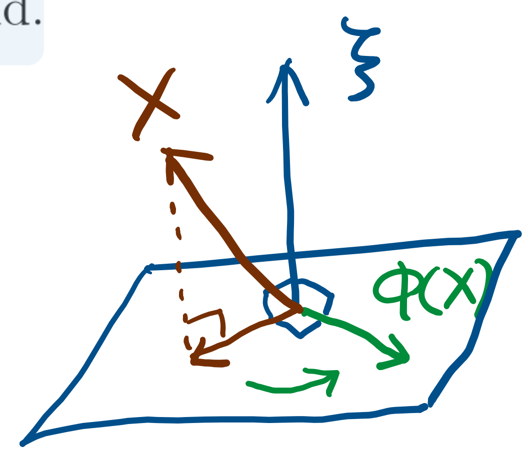
$$g(\xi, \xi) = 1$$

$$g(\phi(\xi), \phi^2(X)) = g(\xi, \phi(X)) - \eta(\xi)\eta(\phi(X)) = 0$$

$$g(\xi, \phi(X)) = 0$$

$$g(\phi(X_\perp), \phi^2(X_\perp)) = g(X_\perp, \phi(X_\perp))$$

$$= g(\phi(X_\perp), -X_\perp) = -g(X_\perp, \phi(X_\perp)) = 0$$



Contact metric manifold

Introduce a (1,1) tensor ϕ : (vector) $\to$ (vector) satifying

$$\phi^2(X) = -X + \eta(X)\,\xi,$$ for arbitrary X.

If g and ϕ satisfy

$$g(\phi(X), \phi(Y)) = g(X, Y) - \eta(X)\eta(Y),$$ for arbitrary X, Y,

$(\mathcal{M}, \phi, \xi, \eta, g)$ is called an almost contact metric manifold.

Furthermore, if it holds that

$$g(X, \phi(Y)) = d\eta(X, Y),$$

$$g_{ab}X^a\phi^b{}_cY^c = (d\eta)_{ac}X^aY^c$$

$$(g_{ab}\phi^b{}_c - (d\eta)_{ac})X^aY^c = 0$$

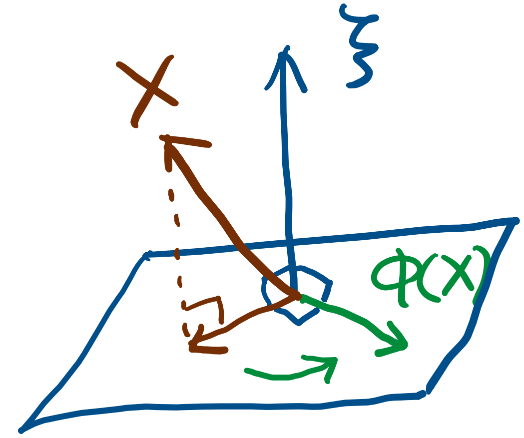
$$g_{ab}\phi^b{}_c = (d\eta)_{ac}$$

$(\mathcal{M}, \phi, \xi, \eta, g)$ is called a contact metric manifold.

From

$$\eta_a \xi^a = 1, \quad d\eta_{ab}\xi^a = 0,$$

$$g_{ab}{\phi^b}_c = (d\eta)_{ac}, \quad g_{ab}{\phi^a}_c{\phi^b}_d = g_{cd} - \eta_c\eta_d$$



we have

$$\phi(\xi) = 0, \qquad {\phi^a}_b\xi^b = (\phi\xi)^a = 0,$$

$$g(X, \phi(X)) = 0, \qquad g_{ab}X^a(\phi X)^b = 0,$$

$$\eta(X) = g(X, \xi), \qquad \boxed{\eta_a = g_{ab}\xi^b}$$

$$g(\xi, \xi) = 1, \qquad \boxed{g_{ab}\xi^a\xi^b = 1}$$

Connection

Levi-Civita connection, covariant derivative ∇

$$\nabla_a g_{bc} = 0 \qquad \text{:metricity,}$$

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) f = 0 \qquad \text{:torsion free}$$

$$(d\eta)_{ab}\xi^a = 0 \quad \Rightarrow \quad \boxed{\xi^a \nabla_a \xi^b = 0} \ : \text{ geodesic}$$

$$\begin{aligned} 0 = d\eta_{ab}\xi^a = (\partial_a \eta_b - \partial_b \eta_a)\xi^a &= (\nabla_a \eta_b - \nabla_b \eta_a)\xi^a \\ &= (\nabla_a (g_{bc}\xi^c) - \nabla_b (g_{ac}\xi^c))\xi^a \\ &= g_{bc}\xi^a \nabla_a \xi^c - \frac{1}{2}\nabla_b (g_{ac}\xi^c \xi^a) \\ &= g_{bc}\xi^a \nabla_a \xi^c \end{aligned}$$

佐々木多様体

佐々木多様体の定義には同値なものがいくつかある.

例えば,
　錐多様体をとったらケーラー多様体になる
といったものが良く知られている.

接触計量多様体が「正規」であるという条件

$$(\nabla_X \phi)Y = g(X, Y) - \eta(Y)X$$

が満たされるとき，この多様体を佐々木多様体という.

このとき，Reeb ベクトルはKillingベクトルになる.

使いやすい条件として次のものもある.

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y$$

3次元Sasaki空間

Contact metric manifold (M, ϕ, ξ, η, g) is a **Sasakian manifold** if

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y,$$

$$\vdots$$

3-dim. Sasaki space is η-Einstein.

$$R^{\mathcal{M}}_{ab} = \beta\ g^{\mathcal{M}}_{ab} + \gamma\ \eta_a \eta_b,$$

3次元Sasaki 空間 $\Leftrightarrow$
接触計量空間で Reeb vector が unit Killing vector

Reeb vector in 3 dim.

$$\eta \wedge d\eta \neq 0 \qquad \Rightarrow \qquad \vec{\xi} \cdot \mathrm{rot}\, \vec{\xi} \neq 0$$

$$(d\eta)_{ab}\xi^a = 0 \qquad \Rightarrow \qquad \xi^a \nabla_a \xi^b = 0$$

$$g_{ab}\xi^a \xi^b = 1$$

The Reeb vector is twisted, and a geodesic tangent.