対称性をもった南部 - 後藤ストリングの古典解

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Introduction

Physical significance of extended objects

- Topological defects e.g. cosmic strings etc
- Braneworld universe model
- ・・・
- AdS/CFT correspondence

Extended objects (cosmic strings) are described by PDE

- Nambu-Goto equation, etc.

Cohomogeneity-one（C-1) objects

C-1 universe

universe models with homogeneous 3-space

e.g. **Friedmann universe model**,

Einstein equations P.D.E. → Friedmann equation O.D.E.

$$ds^2 = -dt^2 + a(t)^2 \left(dx^2 + dy^2 + dz^2\right)$$

C-1 black hole

black holes with spherical symmetry

e.g. **Schwarzschild black hole**

$$ds^2 = -\left(1 - \frac{r_g^2}{r^2}\right) dt^2 + \frac{dr^2}{1 - \frac{r_g^2}{r^2}} + r^2 d\Omega_{S^3}^2$$

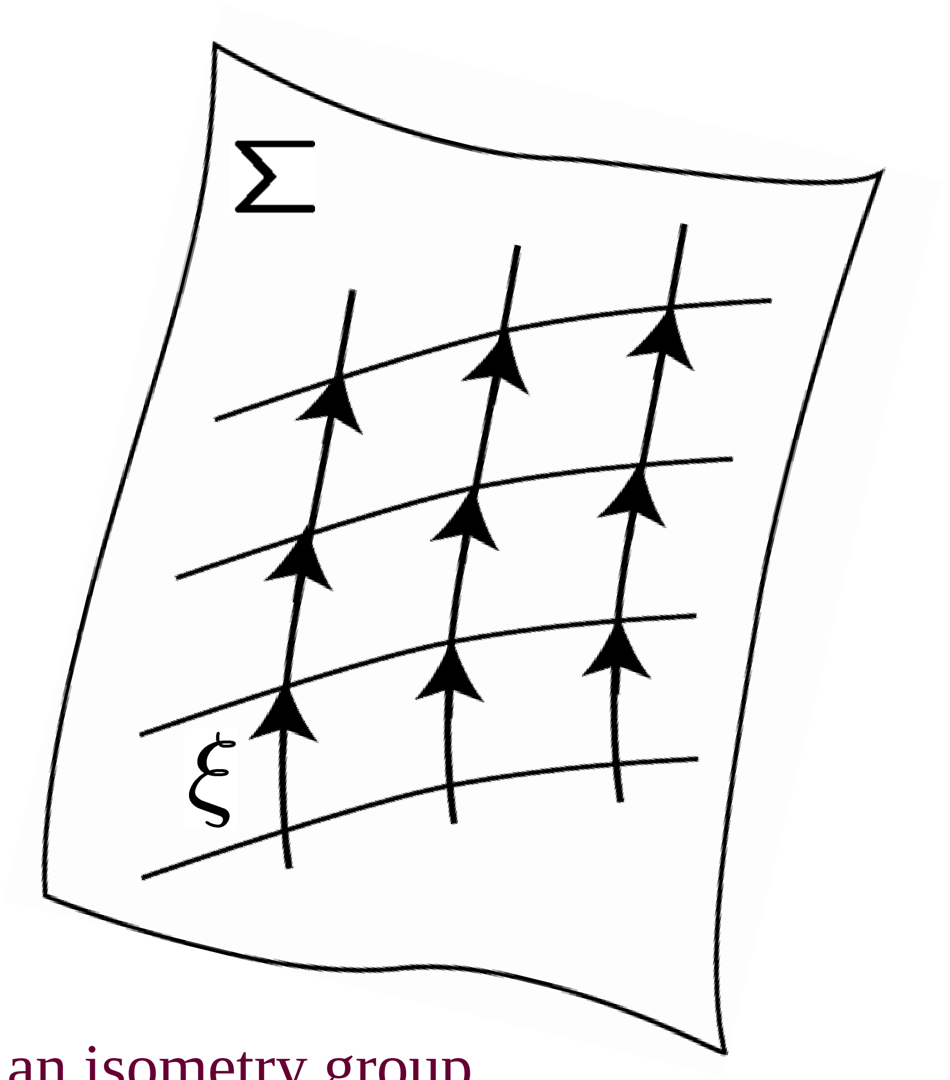
Σ : r=const surface

$$ds_\Sigma^2 = -a_0^2 dt^2 + b_0^2 d\Omega_{S^3}^2$$

C-1 String

1-parameter isometry group
of a target space acts
on the string world sheet

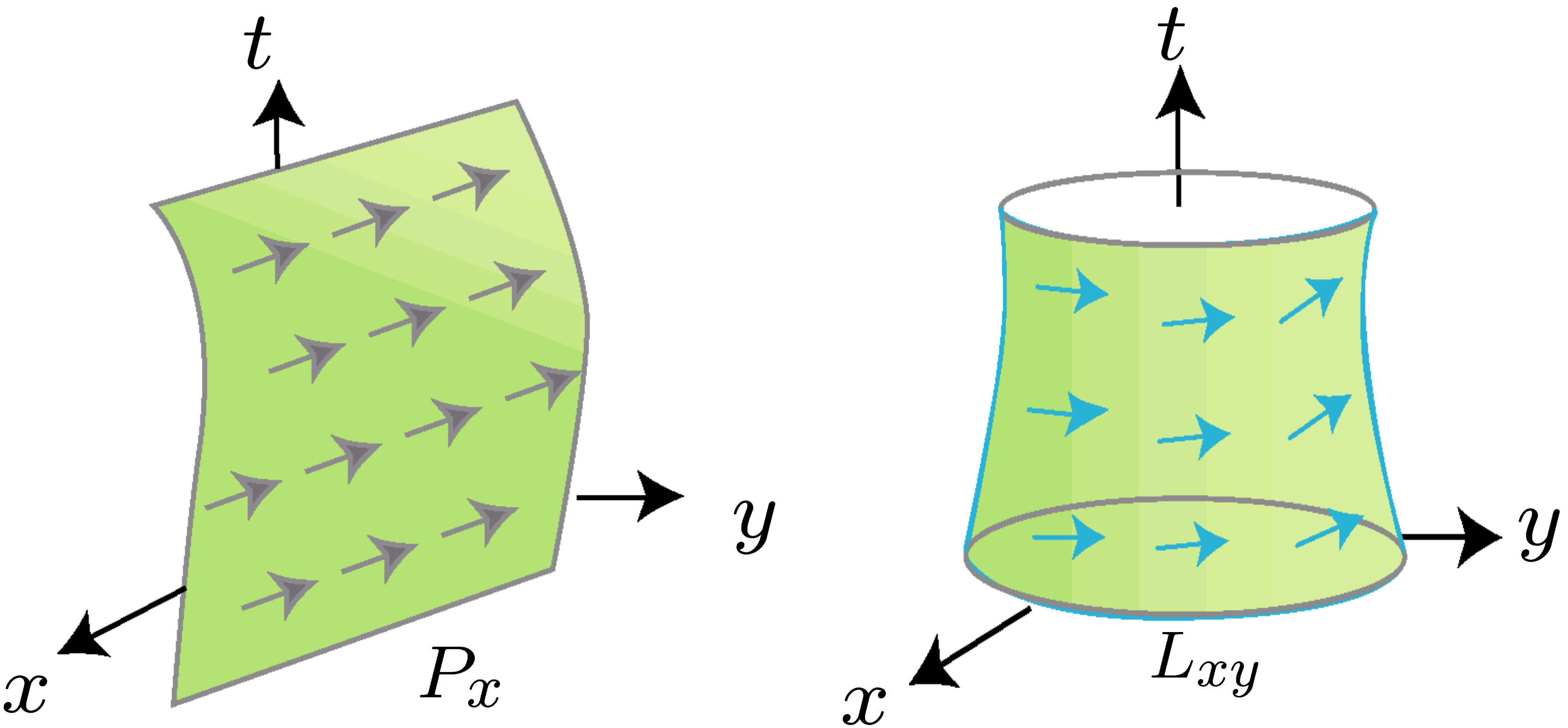
A Killing vector field is
tangent to the world sheet



Killing vector : generator of an isometry group
(infinitesimal version of isometry)

We consider cohomogeneity-one strings

Strings with Symmetry



An isometry acts on the string world sheet.

Advantage of C-1 Objects

Tractable and physically interesting

	Homogeneous	Cohomogeneity-1	No symmetry
To solve	Simplest (algebraic)	Simple (ODE)	Difficult (PDE)
Variety	Poor	Rich	Richest
Physics	Trivial	Non-trivial	General

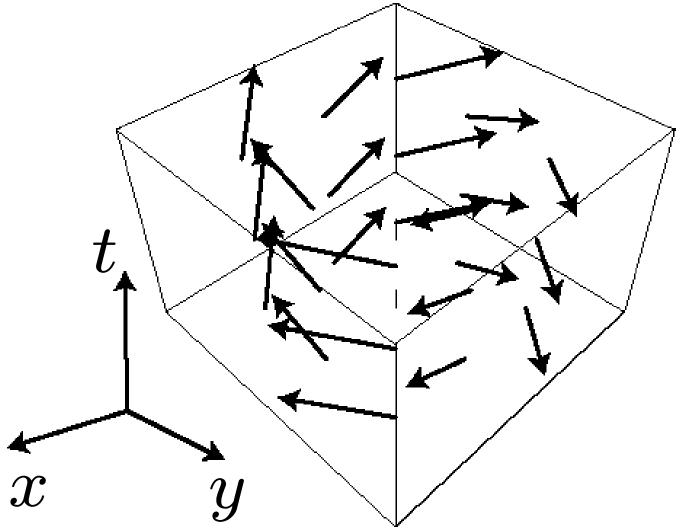
Example 1:
Stationary Rotating Strings in 4D Minkowski

Target space

$$
\begin{aligned}
ds^2 &= g_{\mu\nu}dx^\mu dx^\nu \\
&= -dt^2 + d\rho^2 + \rho^2 d\phi^2 + dz^2
\end{aligned}
$$

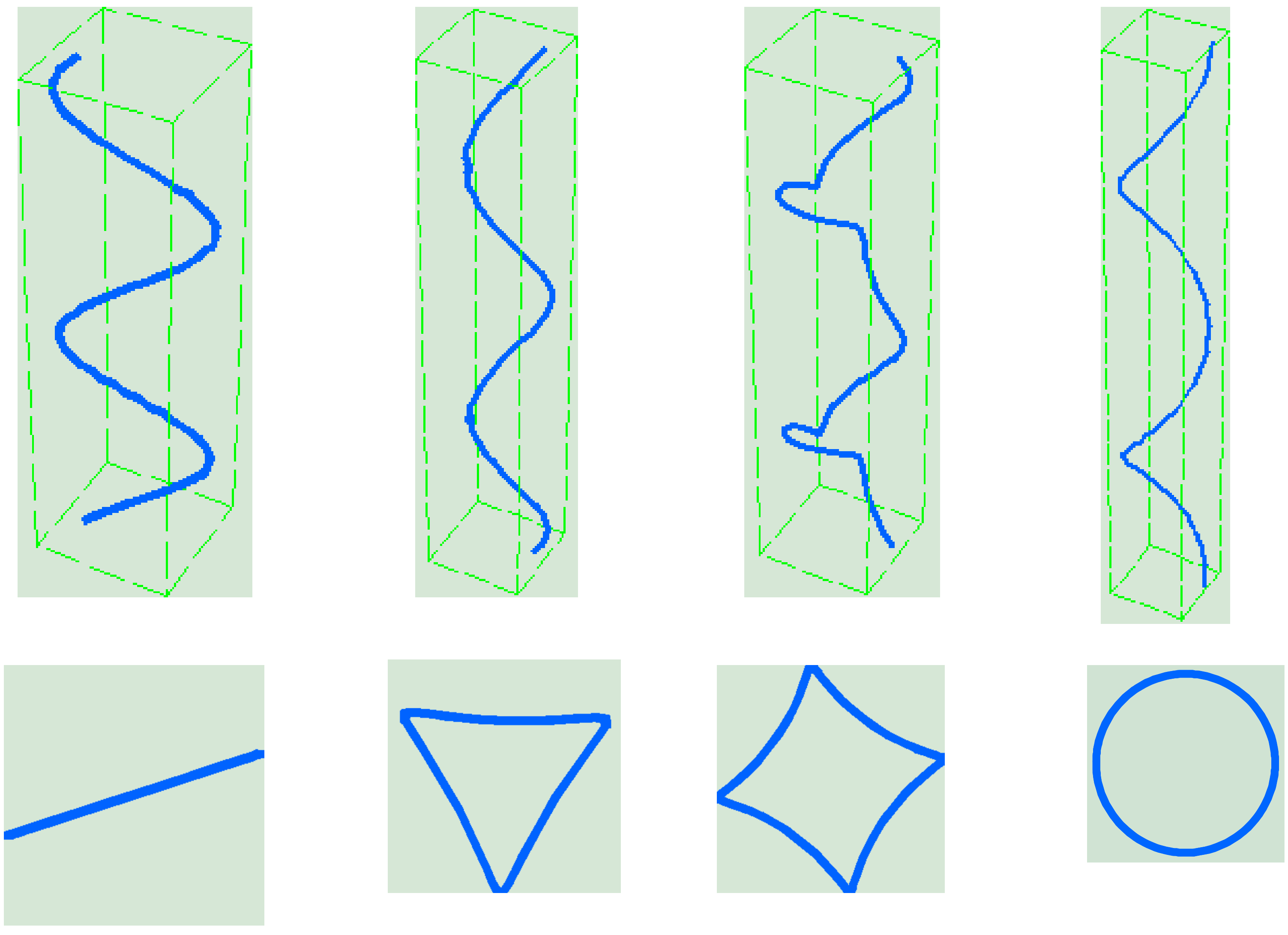
Consider a Killing vector

$$\xi = \partial_t + \Omega \partial_\phi$$



Ogawa, Ishihara, Kozaki, Nakano, Saitoh, PRD78, 023525(2008)

Strings are Rotating



Example 2: Troidal Spirals in 5D Minkowski

$$\begin{aligned} ds^2 &= g_{\mu\nu}dx^\mu dx^\nu \\ &= -dt^2 + d\rho^2 + \rho^2 d\phi^2 + d\zeta^2 + \zeta^2 d\psi^2. \end{aligned}$$

$\partial_t, \partial_\phi, \partial_\psi$ are commutable Killing vectors

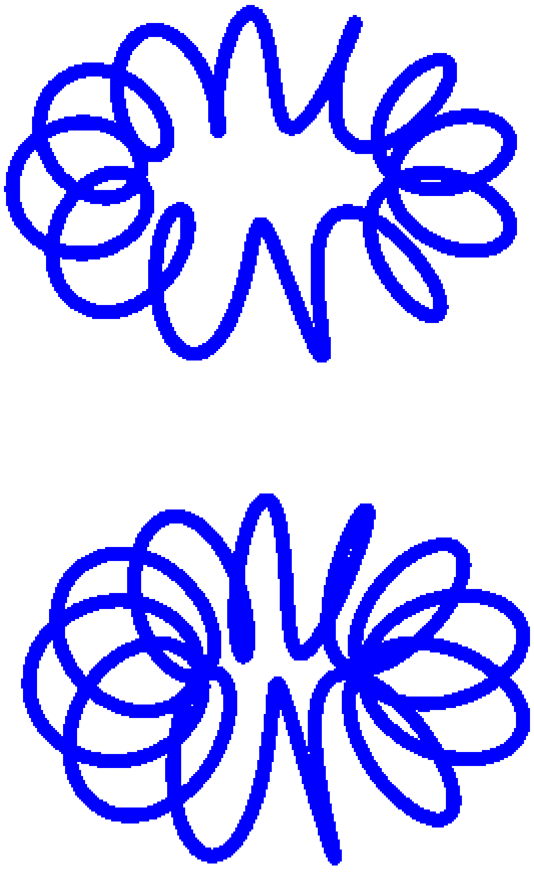
We consider C-1 strings with

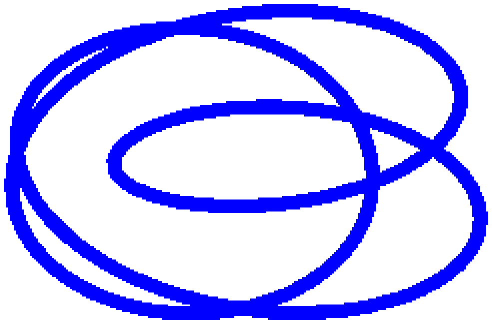
$$\xi = \partial_\phi + \alpha\partial_\psi$$

T. Igata, and H. Ishihara (2010)

T. Igata, H. Ishihara and K.Nishiwaki (2012)

Analytic Solutions

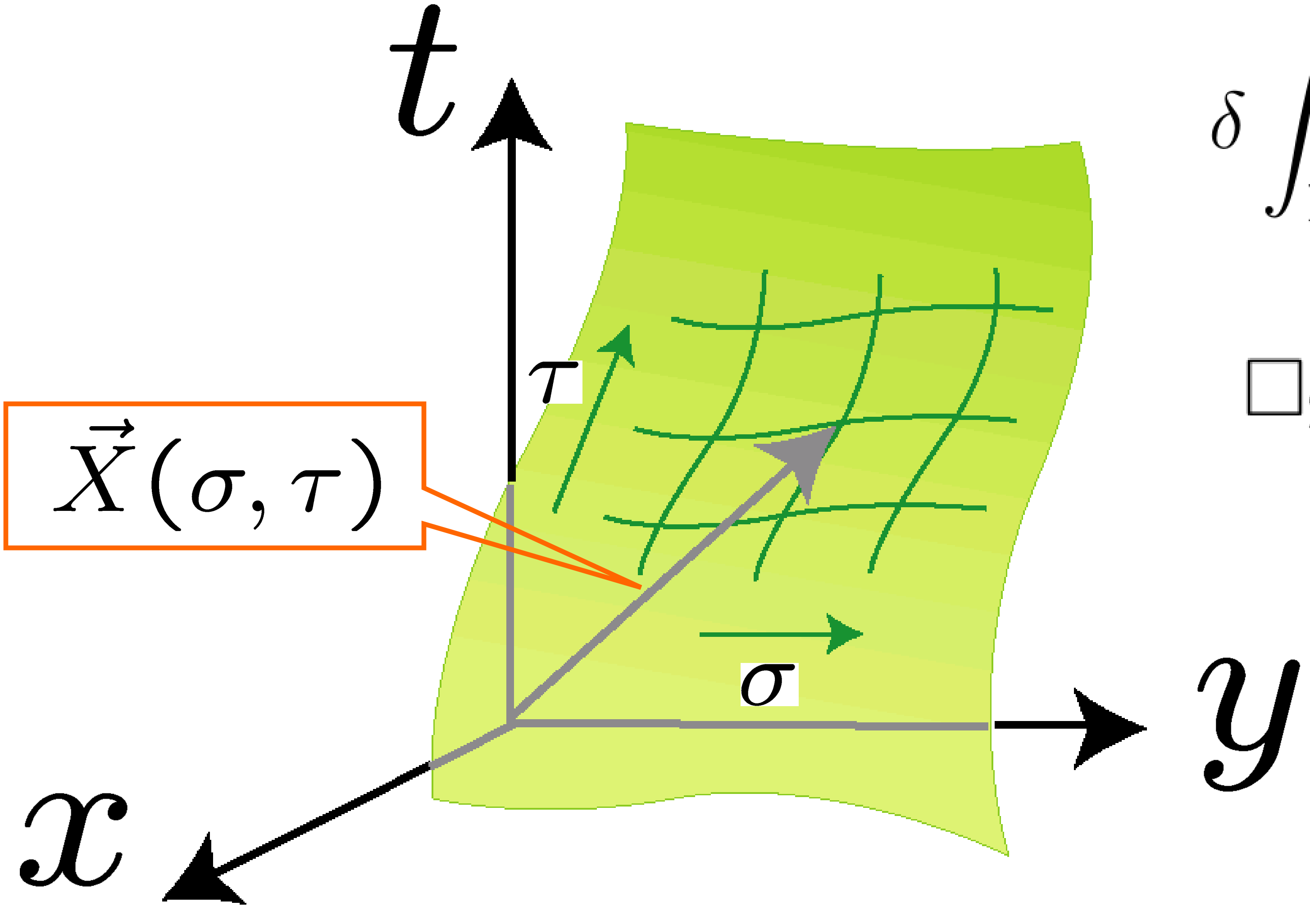




- Cohomogeneity-one strings
  with **timelike or spacelike symmetry**
- Classification of Killing vectors
  (classification of C-1 strings)
- Integrability of equations of motion

Cohomogeneity-one strings

World Sheet



$$\delta \int_{\Sigma} dA = 0$$

$$\Box_2 \vec{X} = 0$$

C-1 String

1-parameter isometry group
of a target space acts
on the world sheet

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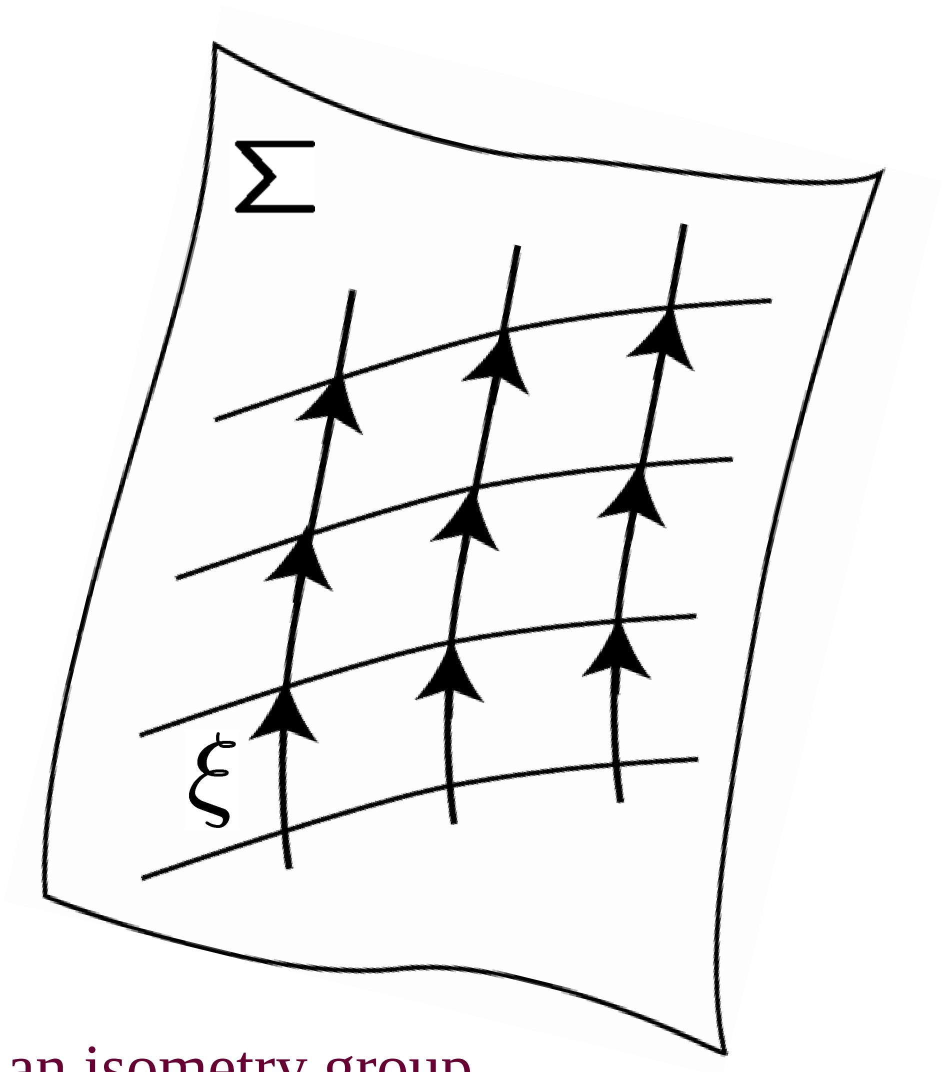
A Killing vector field is
tangent to the world sheet



Killing vector : generator of an isometry group
(infinitesimal version of isometry)

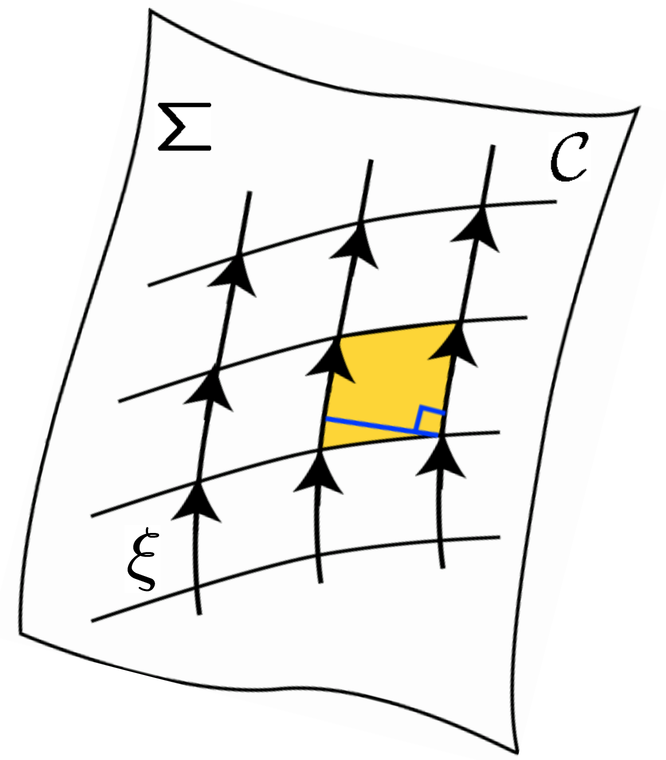
Nambu-Goto action

$$S \propto \int_\Sigma dA = \int_\Sigma |\xi| dl_\perp$$

$$= \int_C \sqrt{(\xi \cdot \xi) h_{\mu\nu} dx^\mu dx^\nu}$$

$$h_{\mu\nu} = g_{\mu\nu} - \frac{\xi_\mu \xi_\nu}{\xi \cdot \xi}$$

We get geodesic action.



Classification of Killing vectors

Equivalence of Killing vectors

Equivalence class of isometry

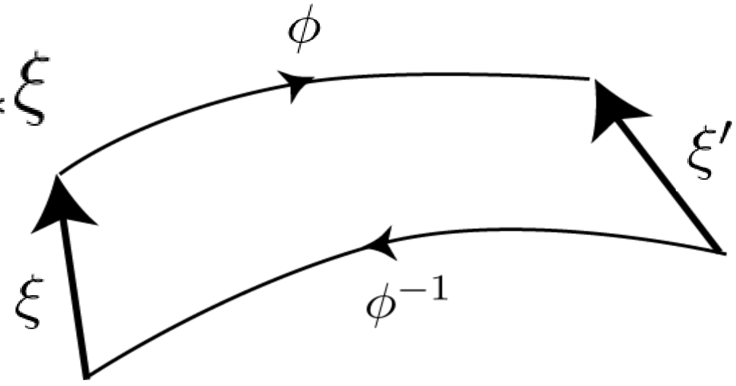
$$g,\ \ g' \in \mathrm{Isom}\mathcal{M} \qquad g \sim g'$$

$$\Leftrightarrow \quad \exists \phi \in \mathrm{Isom}\mathcal{M} \ \text{ s.t. } \ g' = \phi g \phi^{-1}$$

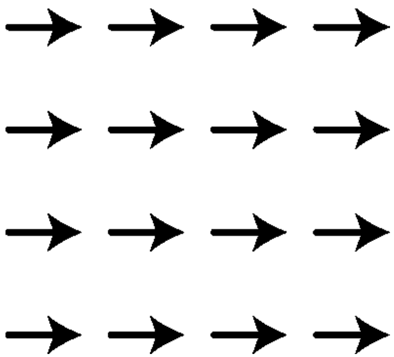
Conjugacy class

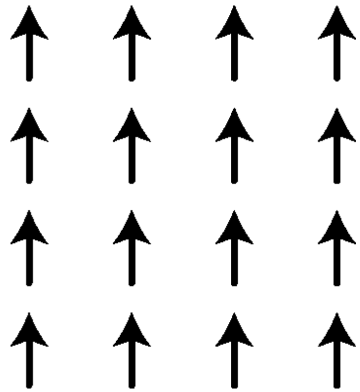
Equivalence of Killing vector

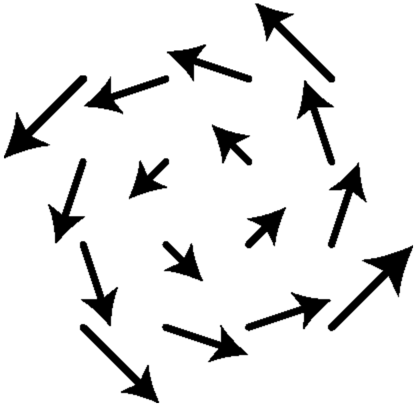
$$\xi \sim \xi' \quad \Leftrightarrow \quad \xi' = \phi_* \xi$$



Isometries in x-y Plane



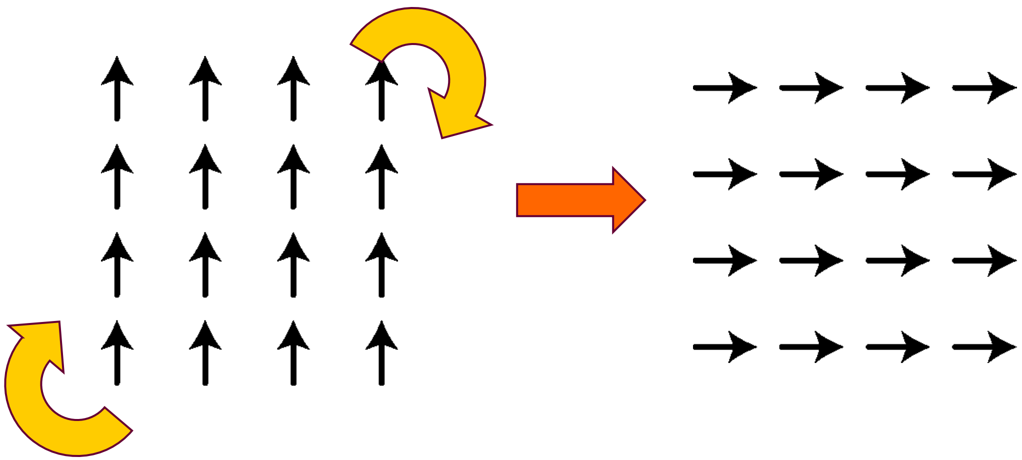




$$P_x = \partial_x \qquad P_y = \partial_y \qquad L_{xy} = x\partial_y - y\partial_x$$

3 linearly independent Klling vector fields

Equivalence Class



$$P_y \sim P_x$$

$$\alpha P_x + \beta P_y \sim P_x$$

$$\alpha P_x + \beta P_y + L_{xy} \sim L_{xy}$$

Equivalence classes $\{P_x, L_{xy}\}$

Classification of Killing vectors

4-dim. Euclid

Type	Canonical form
I	$aP_z + bL_{xy}$
II	$aL_{zw} + bL_{xy}$

4-dim. Minkowski

Type	Canonical form
I	$aP_t + bL_{xy}$
II	$a(P_t + P_z) + bL_{xy}$
III	$aP_z + bL_{xy}$
IV	$aP_z + b(K_{ty} + L_{xy})$
V	$aP_z + bK_{ty}$
VI	$aP_x + b(K_{ty} + L_{xy})$
VII	$aK_{tz} + bL_{xy}$

P : translation

L : rotation

K : Lorentz boost

Ishihara and Kozaki PRD(2005)

Killing Vector Fields in AdS5

Type	Killing vector field
I	$K_{tx} + K_{sy} + L_{xy} + \widetilde{L}_{st} + 2(L_{yz} + K_{tz})$
II	$K_{tx} + \widetilde{L}_{st} + aL_{yz}$
III	$K_{tx} + L_{xy} + aL_{zw}$
IV	$K_{tx} + L_{xy} + aK_{sz}$
V	$K_{tx} + K_{xy} + L_{sw} + L_{wz} + a(L_{xw} - L_{ts} - L_{zy})$
VI	$K_{tx} + K_{sy} + aL_{zw} + b(\widetilde{L}_{st} - L_{xy})$
VII	$K_{tx} + L_{xy} + K_{sy} + \widetilde{L}_{st} + aL_{zw} + b(L_{xy} - \widetilde{L}_{st})$
VIII	$K_{tx} + L_{xy} + K_{sy} + \widetilde{L}_{st} + aL_{zw} + b(K_{ty} + K_{sx})$
IX	$a\widetilde{L}_{st} + bL_{xy} + cL_{zw} \qquad (a^2 + b^2 + c^2 = 1)$
X	$aK_{tx} + bK_{sy} + cL_{zw} \qquad (a^2 + b^2 + c^2 = 1, a \neq \pm b)$

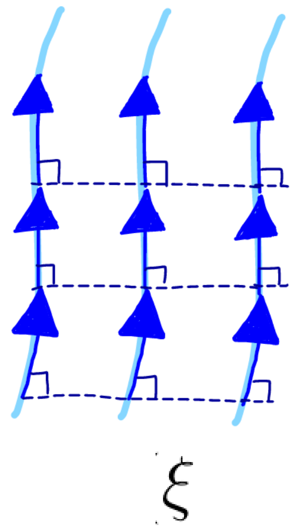
C-1 strings in AdS$_5$ are classified in 10 families.

Classification using SO(4,2) ~ SU(2,2)

T.Koike, H.Kozaki, H.Ishihara, Phys.Rev. D77 (2008) 125003

Orbit space

Geodesics on an orbit space



$$S = \int \sqrt{-\tilde{h}_{ab} dx^a dx^b} \ , \quad \tilde{h}_{ab} = (\xi \cdot \xi)\left(g_{ab} - \frac{\xi_a \xi_b}{\xi \cdot \xi}\right)$$

Classically equivalent

$$S = \frac{1}{2}\int \left(N^{-1}\tilde{h}_{ab}\frac{dx^a}{d\lambda}\frac{dx^b}{d\lambda} - N\right) d\lambda$$

$$= \int (p_a \dot{x}^a - NH)\, d\lambda \ , \qquad H = \frac{1}{2}(\tilde{h}^{ab} p_a p_b + 1)$$

Hamilton equations

$$\dot{x}^a = N\{H, x^a\}$$

$$\dot{p}^a = N\{H, p^a\}$$

$$H \approx 0$$

Geodesic equations

Can we integrate a C-1 string?

Can we integrate geodesics
in the orbit space of a Killing vector?

The system with the degree of freedom N
is integrable in Liouville's sence,
if the number of constants of motion is N.

$\tilde{h}_{ab}$が十分な数のLie可換なKilling vector を許せば
十分な数のPoisson可換な保存量が存在する. ⇒ 積分可能

対称性の高いg_{ab}を target space とすればg_{ab} は
多くのKilling vector を許す.

$$\tilde{h}_{ab} = (\xi \cdot \xi) \left(g_{ab} - \frac{\xi_a \xi_b}{\xi \cdot \xi} \right)$$

ξと可換なg_{ab} のKilling vector は $\tilde{h}_{ab}$ のKilling vector になる.

e^{ξ}を中心とするg_{ab}の isometry の中心化群は $\tilde{h}_{ab}$ のisometry.
その代数の可換な最大部分代数の次元を調べる.

Symmetry of orbit space

4-dim. Minkowski spacetime

Type	Killing vector field	# of commutable KV
I	$aP_t + bL_{xy}$	2
II	$a(P_t + P_z) + bL_{xy}$	2
III	$aP_z + bL_{xy}$	2
IV	$aP_z + b(K_{ty} + L_{xy})$	2
V	$aP_z + bK_{ty}$	2
VI	$aP_x + b(K_{ty} + L_{xy})$	2
VII	$aK_{tz} + bL_{xy}$	1

足りない！

5-dim. Anti-de Sitter spacetime

Type	Killing vector field	# of commutable KV
I	$K_{tx} + K_{sy} + L_{xy} + \widetilde{L}_{st} + 2(L_{yz} + K_{tz})$	3
II	$K_{tx} + \widetilde{L}_{st} + aL_{yz}$	2
III	$K_{tx} + L_{xy} + aL_{zw}$	2
IV	$K_{tx} + L_{xy} + aK_{sz}$	2
V	$K_{tx} + K_{xy} + L_{sw} + L_{wz} + a(L_{xw} - L_{ts} - L_{zy})$	2
VI	$K_{tx} + K_{sy} + aL_{zw} + b(\widetilde{L}_{st} - L_{xy})$	2
VII	$K_{tx} + L_{xy} + K_{sy} + \widetilde{L}_{st} + aL_{zw} + b(L_{xy} - \widetilde{L}_{st})$	2
VIII	$K_{tx} + L_{xy} + K_{sy} + \widetilde{L}_{st} + aL_{zw} + b(K_{ty} + K_{sx})$	2
IX	$a\widetilde{L}_{st} + bL_{xy} + cL_{zw} \quad (a^2 + b^2 + c^2 = 1)$	2
X	$aK_{tx} + bK_{sy} + cL_{zw} \quad (a^2 + b^2 + c^2 = 1, a \neq \pm b)$	2

足りない！

Results

- In the Minkowski spacetime, and 5-dimensional anti-de Sitter spacetime,

  All possible orbit spaces with the metric

$$h_{ab} = |\xi \cdot \xi| \left(g_{ab} - \frac{\xi_a \xi_b}{\xi \cdot \xi} \right)$$

  are geodesically integrable thanks to a

  Killing tensor

  in addition to Killing vectors.

List of Killing tensors

II: $\xi = L_{xt} - L_{st} - aL_{yz}$

$$K_2 = (L_{xy} + L_{ys})^2 + (L_{xz} + L_{zs})^2 + a^2(L_{xw}^2 - L_{xs}^2 - L_{xt}^2 - L_{ws}^2 - L_{wt}^2 + L_{st}^2)$$

$$K_0 = -a^2(x-s)^2$$

V: $\xi = L_{xt} + L_{ys} - L_{yz} \mp L_{xw} + a(L_{xy} + L_{st} \mp L_{zw})$

$$K_2 = (L_{xz} - L_{xs} \pm L_{yw} - L_{yt})^2 + (\pm L_{xw} - L_{xt} - L_{yz} + L_{ys})^2 + 4a[(L_{xz} - L_{xs} \pm L_{yw} - L_{yt})(L_{zs} \mp L_{wt}) + (\pm L_{xw} - L_{xt} - L_{yz} + L_{ys})(L_{zt} \pm L_{ws})]$$

$$K_0 = 4a^2[(z-s)^2 + (w \mp t)^2]$$

X: $\xi = aL_{xt} + bL_{ys} + cL_{zw}$

$$K_2^{(1)} = (b^2 + c^2)\left(L_{xy}^2 - L_{xs}^2 - L_{yt}^2 + L_{st}^2\right) + (a^2 - b^2)\left(-L_{yz}^2 - L_{yw}^2 + L_{zs}^2 + L_{ws}^2\right)$$

$$K_0^{(1)} = (a^2 - b^2)(b^2 + c^2)(y^2 - s^2)$$

- Cohomogeneity-one strings
  with **timelike or spacelike symmetry**
- Classification of Killing vectors
  (classification of C-1 strings)
- Integrability of equations of motion

- Cohomogeneity-one strings
  with **null symmetry**
  → **contact structure**

Spacetime metric admitting a null Killing vector

$$ds^2 = 2\eta_i(x)\,\mathrm{d}x^i\,\mathrm{d}v + h_{ij}(x)\,\mathrm{d}x^i\,\mathrm{d}x^j\ ,$$

$$k = \frac{\partial}{\partial v}\,, \quad g(k,k) = 0, \quad \mathcal{L}_k g = 0$$

: null Killing vector

$$g(k,\cdot) = g(\partial_v,\cdot) = \eta_i dx^i$$

$$k_\mu = g_{\mu\nu}k^\nu = \eta_\mu$$

$$g(k,k) = \eta(k) = 0, \quad \boxed{\iota_k\eta = 0}$$

: null condition

Spacetime metric admitting a null Killing vector

$$ds^2 = 2\eta_i(x)\, dx^i\, dv + \boxed{h_{ij}(x)\, dx^i\, dx^j}$$

$$k = \frac{\partial}{\partial v}\,, \quad g(k,k) = 0, \quad \mathcal{L}_k g = 0$$

: null Killing vector

$$g(k,\cdot) = g(\partial_v,\cdot) = \eta_i dx^i$$

$$k_\mu = g_{\mu\nu}k^\nu = \eta_\mu$$

$$g(k,k) = \eta(k) = 0, \quad \iota_k \eta = 0$$: null condition

(Σ_0, h) : 3-dimensional space

1-form $\eta = \eta_i(x) dx^i$ is naturally induced on Σ_0.

null Killing vector field

$$\nabla_{(\mu}k_{\nu)} = 0, \qquad g_{\mu\nu}k^\mu k^\nu = 0.$$

$$\nabla_\nu k_\mu = \nabla_{(\nu}k_{\mu)} + \nabla_{[\nu}k_{\mu]} = \frac{1}{2}(k_{\mu,\nu} - k_{\nu,\mu}) = \frac{1}{2}\,\mathrm{d}\eta_{\nu\mu}\,,$$

$$0 = k^\mu(\nabla_\mu k_\nu + \nabla_\nu k_\mu) = k^\mu \nabla_\mu k_\nu + \frac{1}{2}\nabla_\nu(k^\mu k_\mu)$$

$$= k^\mu \nabla_\mu k_\nu$$

$$k^\nu \nabla_\nu k^\mu = 0, \qquad \iota_k\,\mathrm{d}\eta = 0$$

: geodesic tangent

Null Killing vector k is a null geodesic tangent.

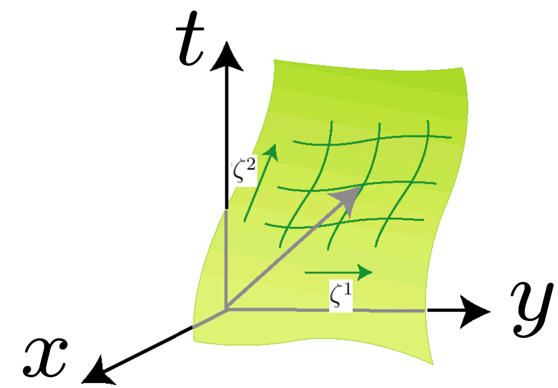
$$\iota_k \eta = 0$$

$$\iota_k d\eta = 0$$

Nambu-Goto action

$$S = \int \sqrt{-\gamma}\, \mathrm{d}\zeta^1\, \mathrm{d}\zeta^2\,, \qquad \gamma := \det \gamma_{ab}.$$

$$\gamma_{ab} = g_{\mu\nu} \frac{\partial x^\mu}{\partial \zeta^a} \frac{\partial x^\nu}{\partial \zeta^b}.$$



Equation of motion

$$\frac{\partial}{\partial \zeta^a}\left(\sqrt{-\gamma}\, \gamma^{ab} \frac{\partial x^\mu}{\partial \zeta^b}\right) + \sqrt{-\gamma}\, \gamma^{ab} \Gamma^\mu{}_{\nu\lambda} \frac{\partial x^\nu}{\partial \zeta^a} \frac{\partial x^\lambda}{\partial \zeta^b} = 0,$$

: Nonlinear P.D.E.

Null coordinates on a string world sheet

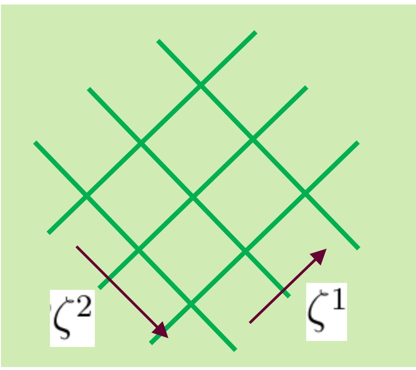
$$\mathrm{d}s^2 = \gamma_{ab}\, \mathrm{d}\zeta^a\, \mathrm{d}\zeta^b = 2\, \gamma_{12}(\zeta^1, \zeta^2)\, \mathrm{d}\zeta^1\, \mathrm{d}\zeta^2\,,$$

Equation of motion

$$\frac{\partial x^\nu}{\partial \zeta^2} \nabla_\nu \frac{\partial x^\mu}{\partial \zeta^1} = 0,$$

⬌

$$\frac{\partial}{\partial(x-t)} \frac{\partial}{\partial(x+t)} f(t,x) = 0$$



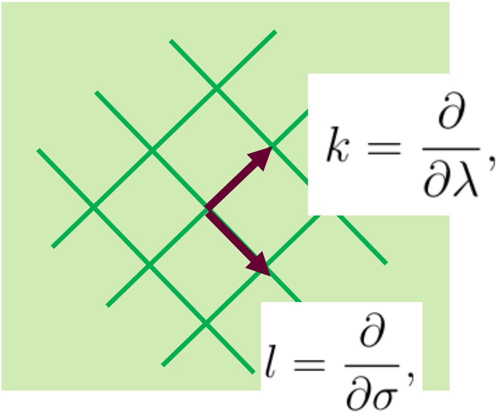
Null coordinate system $(\zeta^1 = \lambda, \zeta^2 = \sigma)$

on the string world sheet $\mathcal{S}$.

$$ds_{\mathcal{S}}^2 = \gamma_{ab} d\zeta^a d\zeta^b = 2\gamma_{\lambda\sigma}(\lambda, \sigma) d\lambda d\sigma$$

$k = \dfrac{\partial}{\partial \lambda},$ null Killing vector

$l = \dfrac{\partial}{\partial \sigma},$ null vector



$$ds_{\mathcal{S}}^2 = 2\gamma_{\lambda\sigma}(\sigma) d\lambda d\sigma = 2 d\lambda d\tilde{\sigma} \to 2 d\lambda d\sigma$$

$\gamma_{\lambda\sigma} = g(k, l) = 1,$ ➡ $\boxed{\iota_l \eta = 1,}$

$\gamma_{\lambda\lambda} = g(k, k) = 0, \quad \gamma_{\sigma\sigma} = g(l, l) = 0,$

$\dfrac{\partial x^\nu}{\partial \zeta^2} \nabla_\nu \dfrac{\partial x^\mu}{\partial \zeta^1} = 0,$ ➡ $l^\mu \nabla_\mu k_\nu = 0$ ➡ $\boxed{\iota_l d\eta = 0}$

$$ds^2_{\mathcal{S}} = 2d\lambda d\sigma$$

$$(l \cdot \nabla)k = 0$$

$$\gamma_{\lambda\sigma} = g(k, l) = 1,$$

$$\iota_l \, \mathrm{d}\eta = 0,$$

$$\iota_l \eta = 1.$$

On a 4-dim. spacetime

$$\iota_l \eta = 1, \quad \iota_l d\eta = 0, \quad \iota_k \eta = 0, \quad \iota_k d\eta = 0$$

Two cases

$(1) d\eta \neq 0$

$(2) d\eta = 0$

On a 4-dim. spacetime

$$\iota_l \eta = 1, \quad \iota_l d\eta = 0, \quad \iota_k \eta = 0, \quad \iota_k d\eta = 0$$

In the case

$$(1) d\eta \neq 0$$

$$\iota_l[\eta \wedge (\mathrm{d}\eta)] = (\iota_l \eta) \wedge (\mathrm{d}\eta) - \eta \wedge [\iota_l(\mathrm{d}\eta)]$$
$$= \iota_l \eta \, (\mathrm{d}\eta) \neq 0,$$

$$\eta \wedge d\eta \neq 0$$

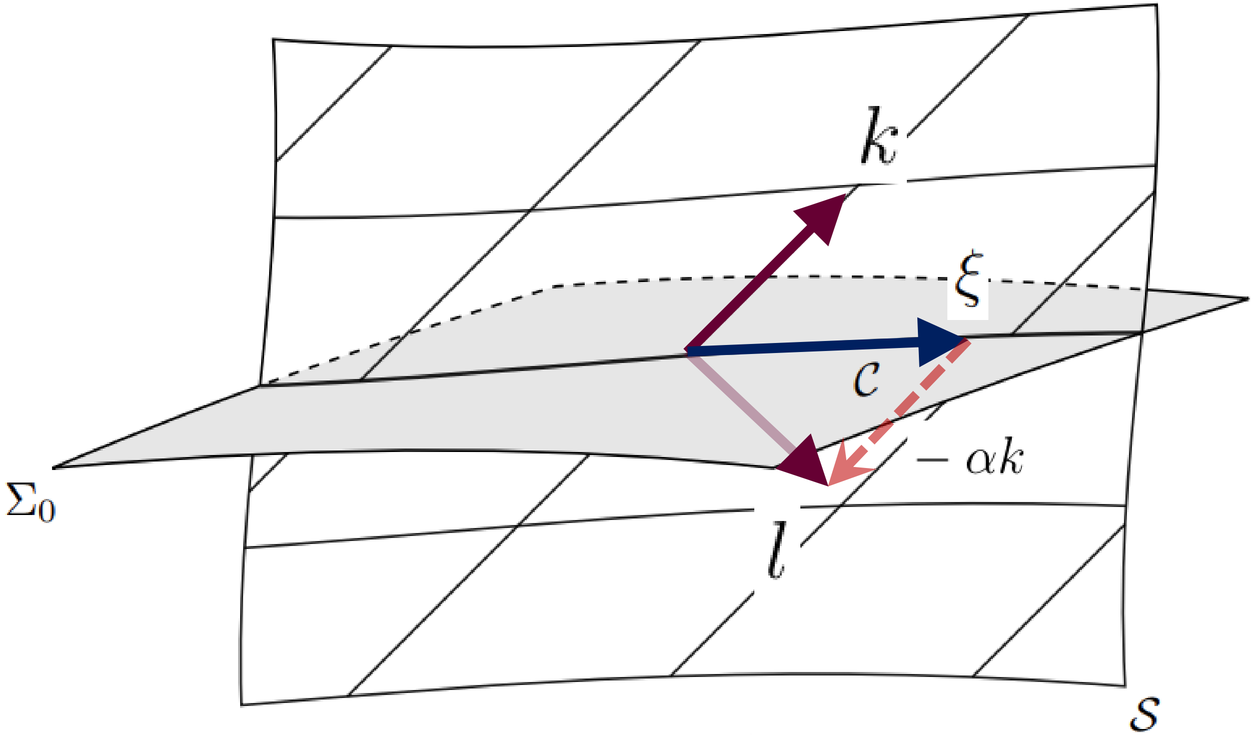
Σ_0 has a contact structure.

The Reeb vector ξ, which satisfies

$$\iota_\xi \eta = 1, \quad \iota_\xi d\eta = 0,$$

:Reeb vector

is uniquely determined on Σ_0.



$$l = \xi - \alpha k$$

$$
\begin{aligned}
g(l,l) &= g(\xi - \alpha k, \xi - \alpha k) \\
&= g(\xi,\xi) - 2\alpha g(\xi,k) + \alpha^2 g(k,k) \\
&= 0
\end{aligned}
$$

$$\alpha = \frac{1}{2}\frac{g(\xi,\xi)}{g(\xi,k)}$$

On a 4-dim. spacetime with a null Killing vector

$$\iota_l \eta = 1, \quad \iota_l d\eta = 0, \quad \iota_k \eta = 0, \quad \iota_k d\eta = 0$$

In the case $(1) d\eta \neq 0$

On the 3-dim. space Σ_0

$$l = \xi - \alpha k$$

$$\iota_\xi \eta = 1, \quad \iota_\xi d\eta = 0,$$

$$\iota_l \eta = \iota_{(\xi - \alpha k)} \eta = \iota_\xi \eta - \alpha \iota_k \eta$$

$$= \iota_\xi \eta$$

$$\iota_l d\eta = \iota_{(\xi - \alpha k)} d\eta = \iota_\xi d\eta - \alpha \iota_k d\eta$$

$$= \iota_\xi d\eta$$

$l = \xi - \alpha k$ is a solution.

On a 4-dim. spacetime with a null Killing vector

$$\iota_l \eta = 1, \quad \iota_l d\eta = 0, \quad \iota_k \eta = 0, \quad \iota_k d\eta = 0$$

In the case $(1) d\eta \neq 0$

On the 3-dim. space Σ_0

$$l = \xi - \alpha k$$

$$\iota_\xi \eta = 1, \quad \iota_\xi d\eta = 0,$$

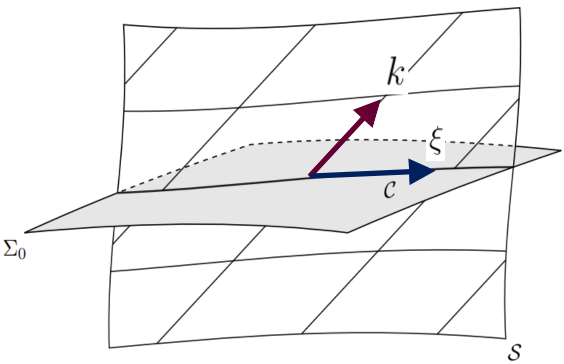
$$\iota_l \eta = \iota_{(\xi - \alpha k)} \eta = \iota_\xi \eta - \alpha \iota_k \eta$$

$$= \iota_\xi \eta$$

$$\iota_l d\eta = \iota_{(\xi - \alpha k)} d\eta = \iota_\xi d\eta - \alpha \iota_k d\eta$$

$$= \iota_\xi d\eta$$

An integral curve of the Reeb vector ξ determines $\mathcal{S}$ completely.



On a 4-dim. spacetime

$$\iota_l \eta = 1, \quad \iota_l d\eta = 0, \quad \iota_k \eta = 0, \quad \iota_k d\eta = 0$$

In the case $\boxed{(2) d\eta = 0}$

Equation of motion $\iota_l d\eta = 0$ is trivial.

If ξ on Σ_0 satisfies $\iota_\xi \eta = 1$,

$l = \xi - \alpha k$ satisfies $\iota_l \eta = 1$.

ξ is not unique.

$$\eta = dw, \quad \xi = \frac{\partial}{\partial w} + f^i(x^j) \frac{\partial}{\partial x^i}$$

$f^i(x^j)$: arbitrary functions.

Example: Einstein's static universe

$$\mathrm{d}s^2 = a^2\left[-\mathrm{d}t^2 + \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2 + (\mathrm{d}\psi - \cos\theta\,\mathrm{d}\phi)^2\right].$$

null Killing vector field

$$k = \frac{\partial}{\partial t} + \frac{\partial}{\partial \psi}$$

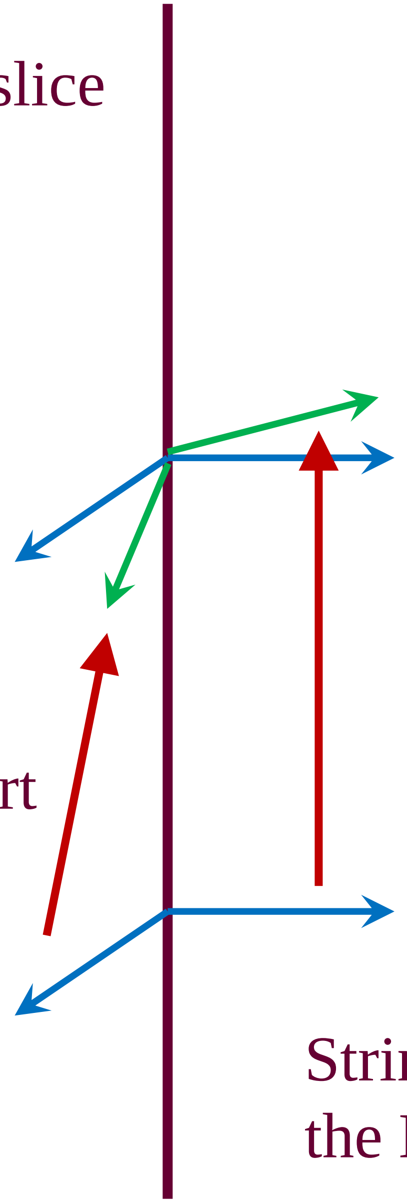
$$ds^2 = 2a^2(d\psi - \cos\theta d\phi)dv + a^2\left(d\theta^2 + \sin^2\theta d\phi^2 + (d\psi - \cos\theta d\phi)^2\right)$$

$$\eta = a^2(d\psi - \cos\theta d\phi)$$

$$\xi = \frac{1}{a^2}\frac{\partial}{\partial\psi} \longrightarrow l = \frac{1}{a^2}\left(-\frac{\partial}{\partial t} + \frac{\partial}{\partial\psi}\right)$$

$\mathcal{S}$ is spanned by k and l,

equivalently, spanned by $\dfrac{\partial}{\partial t}$ and $\dfrac{\partial}{\partial\psi}$.



String with null symmetry in the Einstein static universe is **twisted.**

C1 strings with null symmetry is closely related to the contact structure.