4. 接触構造を用いた アインシュタイン方程式の解

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} 現象

Introduction

電磁気力と重力

電磁気力：　Maxwell方程式　　（線形）

$$\partial_a F^{ab} = \sum e u^b$$
$$m\frac{d}{d\tau}u^a = eF^a{}_b u^b$$

電磁場と電流を同時に決定するのは難しい

重力：　　　Einstein方程式　　（非線形）

重力場を曲がった時空としての幾何学的な記述

重力の基礎方程式

Einstein方程式

Riemann曲率のトレース部分である
Ricci曲率と物質のエネルギーを結びつける

$$R_{ab} = T_{ab} - \frac{1}{2}T g_{ab} + \Lambda g_{ab},$$

$$\nabla_a T^{ab} = 0$$

重力場と物質の運動を同時に決定するのは難しい

Exact solutions to the Einstein equation

- Vacuum solutions $R_{ab} = 0$
  - Minkowski spacetime
  - Schwarzshild BH, Kerr BH
  - $\cdots$
  - de Sitter, Anti-de Sitter, (with a cosmological constant)
  - $\cdots$
- Solutions with matter $R_{ab} = T_{ab} - \frac{1}{2} T g_{ab} + \Lambda g_{ab},$
  - Friedmann universe (perfect fluid)
  - Einstein's static universe (perfect fluid and cosmological const.) $R^1 \times S^3$
  - Bertotti-Robinson (magnetic field) $AdS^2 \times S^2$
  - Tolemann-Bondi (dust fluid) with arbitrary functions
  - $\cdots$

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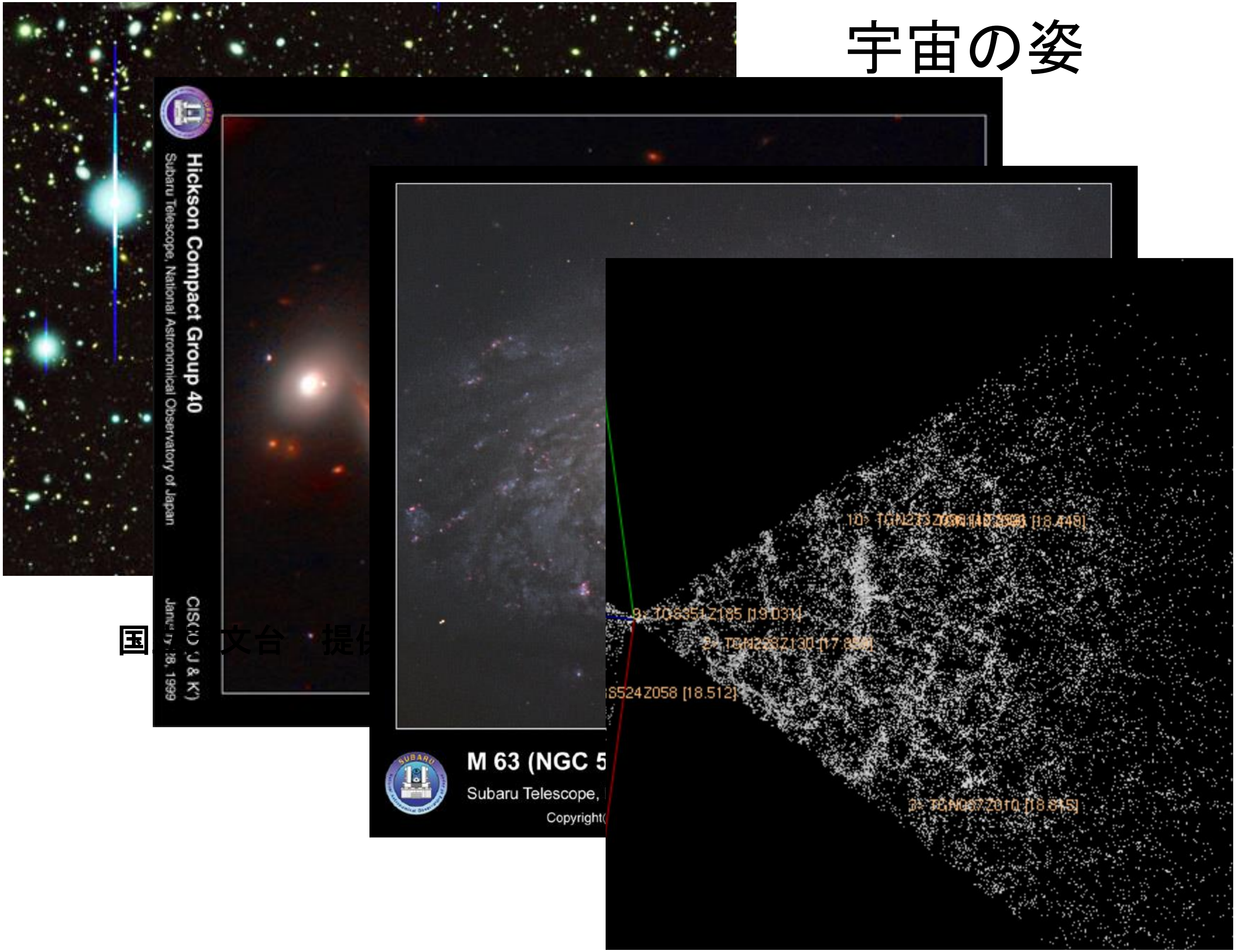
PTEP 2022 (2022) 2, 023E01

宇宙の姿


Hickson Compact Group 40
Subaru Telescope, National Astronomical Observatory of Japan

目的

非一様性な物質を源とした
Einstein方程式の厳密解を構成する

Spacetime consists of

$R^1 \times$ (3-dim. Sasaki space) : contact metric space

3-dim. Sasaki space is η-Einstein.

$$R^{\mathcal{M}}_{ab} = \beta \ g^{\mathcal{M}}_{ab} + \gamma \ \eta_a \eta_b,$$

η : Contact 1-form

物質も η で特徴づけられていれば，全体として解が構成できる

metric

$$ds^2 = -dt^2 + ds_M^2,$$

$$ds_M^2 = a^2\left(d\theta^2 + h(\theta,\phi)^2\ d\phi^2\right) + b^2(d\psi + f(\theta,\phi)\ d\phi)^2,$$

$$h(\theta,\phi) := \partial_\theta f(\theta,\phi).$$

$$\sigma^0 := dt, \quad \sigma^1 := a d\theta, \quad \sigma^2 := a h(\theta,\phi) d\phi, \quad \sigma^3 := b(d\psi + f(\theta,\phi)\ d\phi),$$

$$= \eta$$

$$d\sigma^3 = \frac{b}{a^2}\sigma^1 \wedge \sigma^2, \qquad \sigma^3 \wedge d\sigma^3 \neq 0,$$

σ^3 contact form

$$\xi_{(t)} = \partial_t, \qquad \xi_{(\psi)} = \frac{1}{b}\partial_\psi$$

Reeb vector は Killing vector

On *M*

$$ds^2 = -dt^2 + ds_M^2,$$

$$ds_M^2 = a^2\left(d\theta^2 + h(\theta,\phi)^2\ d\phi^2\right) + b^2(d\psi + f(\theta,\phi)\ d\phi)^2,$$

$$h(\theta,\phi) := \partial_\theta f(\theta,\phi).$$

$$\sigma^1 := a d\theta, \quad \sigma^2 := a h(\theta,\phi) d\phi, \quad \sigma^3 := b(d\psi + f(\theta,\phi)\ d\phi), \\ = \eta$$

$$d\eta = \frac{b}{a^2}\sigma^1 \wedge \sigma^2$$

$$^*d\eta = \frac{b}{a^2}\sigma^1 \wedge \sigma^2 = \frac{b}{a^2}\eta$$

$$\mathrm{rot}\ \boldsymbol{\eta} \propto \boldsymbol{\eta}$$

Beltrami vector (1-form) field

Beltrami 場 $\mathrm{rot}\ \boldsymbol{\eta} = \lambda\boldsymbol{\eta}$ に対して

$$\boldsymbol{B} = \boldsymbol{\eta} \quad とと るると,$$

$$\mathrm{rot}\ \boldsymbol{B} = \lambda\boldsymbol{B} = \boldsymbol{J}$$

つまり，電流も $\boldsymbol{\eta}$ で表される．

電流と磁場が同じ向きなので，電流に力が働かない (force free).

Reeb vector $\quad \xi^a = g^{ab}\eta_b$

ξ は geodesic tangent $\quad (\xi\cdot\nabla)\xi = 0$

電流を担う荷電粒子は磁場に沿って測地運動する．

Beltrami vector 場で Maxwell-current system が解ける．

Ricci 曲率

$$ds^2 = -dt^2 + ds_M^2,$$

$$ds_M^2 = \underline{a^2 \left(d\theta^2 + h(\theta,\phi)^2 \ d\phi^2\right)} + b^2 (d\psi + f(\theta,\phi) \ d\phi)^2,$$

N : base space $\qquad h(\theta,\phi) := \partial_\theta f(\theta,\phi).$

Ricci curvature

$$R_{ab} = \left(\frac{1}{2}R_N - \frac{b^2}{2a^4}\right)(\sigma_a^1\sigma_b^1 + \sigma_a^2\sigma_b^2 + \sigma_a^3\sigma_b^3) + \left(\frac{b^2}{a^4} - \frac{1}{2}R_N\right)\sigma_a^3\sigma_b^3,$$

$$R_N = -\frac{2}{a^2}\frac{\partial_\theta^2 h(\theta,\phi)}{h(\theta,\phi)},$$

N の scalar curvature

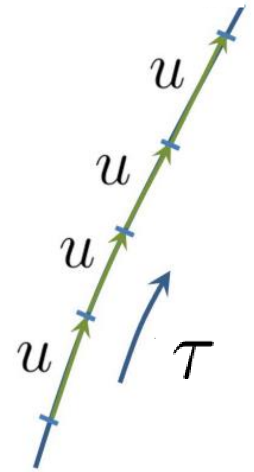
自由粒子の系

4-velocity

$$u^a = \frac{dx^\mu}{d\tau}\partial_\mu \qquad u^t = \frac{cdt}{d\tau} = \frac{c}{\sqrt{1 - v^2/c^2}}$$

Lorentz factor

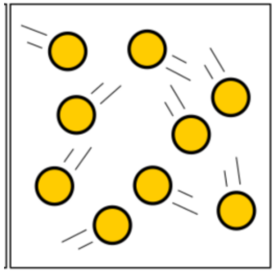
$$g_{ab}u^a u^b = \frac{g_{\mu\nu}dx^\mu dx^\nu}{d\tau^2} = -c^2$$



Energy

$$E = \frac{mc^2}{\sqrt{1 - v^2/c^2}} \simeq mc^2 + \frac{1}{2}mv^2 + \cdots$$

$$\simeq mc^2, \qquad (v \ll c)$$



Energy density

$$\rho = mnu^t u^t$$

n : number density

Energy-momentum tensor

$$T^{ab} = mnu^a u^b$$

熱運動による運動エネルギー（圧力）を無視した多粒子系

ダスト流体

ダスト流体

$$\xi_{(t)} = \partial_t, \quad \xi_{(\psi)} = \partial_\psi,$$

Counter flow　±Reeb vector方向に運動する粒子

$$u_+^a = \frac{1}{\sqrt{1-v^2}}\xi_{(t)}^a + \frac{v}{\sqrt{1-v^2}}\xi_{(\psi)}^a,$$

$$u_-^a = \frac{1}{\sqrt{1-v^2}}\xi_{(t)}^a - \frac{v}{\sqrt{1-v^2}}\xi_{(\psi)}^a,$$

where v is a function on N.

$$u_\pm^a \nabla_a u_\pm^b = 0.$$

粒子の運動方程式が解けた

Energy–momentum tensor

$$T^{ab} = \frac{1}{2}mn(u_+^a \otimes u_+^b + u_-^a \otimes u_-^b)$$

$$= mn\left(\frac{1}{1-v^2}\xi_{(t)}^a \otimes \xi_{(t)}^b + \frac{v^2}{1-v^2}\xi_{(\psi)}^a \otimes \xi_{(\psi)}^b\right),$$

n : number density

Einstein方程式

重力源は，ダスト粒子系，磁場，電流
とすることが可能だが，
ここでは，**ダスト粒子系**だけ考える．

Einstein 方程式

$$R_{ab} = T_{ab} - \frac{1}{2}Tg_{ab} + \Lambda g_{ab},$$

$$R_{ab} = \left(-\frac{b^2}{2a^4} + \frac{1}{2}R_N\right)(\sigma_a^1 \otimes \sigma_b^1 + \sigma_a^2 \otimes \sigma_b^2) + \frac{b^2}{2a^4}\sigma_a^3 \otimes \sigma_b^3.$$

$$T_{ab} = mn\left(\frac{1}{1-v^2/c^2}\sigma_a^0 \otimes \sigma_b^0 + \frac{v^2}{1-v^2/c^2}\sigma_a^3 \otimes \sigma_b^3\right)$$

$$-\frac{1}{2}Tg_{ab} + \Lambda g_{ab}$$

$$= (\Lambda - \frac{1}{2}T)(-\sigma_a^0 \otimes \sigma_b^0 + \sigma_a^1 \otimes \sigma_b^1 + \sigma_a^2 \otimes \sigma_b^2 + \sigma_a^3 \otimes \sigma_b^3)$$

Einstein 方程式

$$R_{ab} = T_{ab} - \frac{1}{2}T g_{ab} + \Lambda g_{ab},$$

(0,0)

$$0 = \frac{1}{2}mn(\theta,\phi)\left(\frac{1+v(\theta,\phi)^2}{1-v(\theta,\phi)^2}\right) - \Lambda,$$

(1,1) (2,2)

$$-\frac{b^2}{2a^4} + \frac{1}{2}R_N = \frac{1}{2}mn(\theta,\phi) + \Lambda,$$

(3,3)

$$\frac{b^2}{2a^4} = \frac{1}{2}mn(\theta,\phi)\left(\frac{1+v(\theta,\phi)^2}{1-v(\theta,\phi)^2}\right) + \Lambda.$$

$$\Lambda = \frac{b^2}{4a^4} > 0$$

N 上のEinstein方程式

$$R_N(\theta, \phi) = mn(\theta, \phi) + 6\Lambda.$$

$$R_N = -\frac{2}{a^2}\frac{\partial_\theta^2 h(\theta, \phi)}{h(\theta, \phi)},$$

線形微分方程式

$$\partial_\theta^2 h(\theta,\phi) + a^2 w(\theta,\phi) h(\theta,\phi) = 0,$$

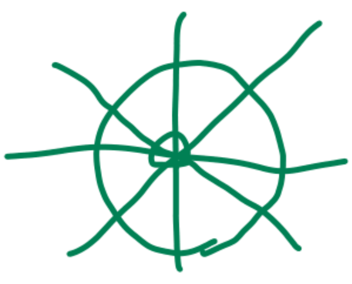
$$w(\theta,\phi) := \frac{1}{2} mn(\theta,\phi) + 3\Lambda.$$

$$ds_M^2 = a^2\left(d\theta^2 + h(\theta,\phi)^2\, d\phi^2\right) + b^2 (d\psi + f(\theta,\phi)\, d\phi)^2,$$

$$h(0,\phi) = 0,$$

$$h'(0,\phi) = 1.$$

$$h(\theta,0) = h(\theta,2\pi)$$



Examples

homogeneous cases

$$\frac{b^2}{4a^4} = \Lambda,$$

$$mn(1 + 2v^2) = 2\Lambda,$$

$$h'' + a^2 w h = 0, \qquad w(\theta, \phi) := \frac{1}{2} mn(\theta, \phi) + 3\Lambda,$$

const.

$$h = \sin\theta, \quad a^2 = \frac{2}{mn + 6\Lambda}. \quad v^2 = \frac{\Lambda}{mn} - \frac{1}{2}.$$

$$f = -\cos\theta,$$

$$ds^2 = -dt^2 + a^2 \left(d\theta^2 + \sin^2\theta \; d\phi^2 + 4a^2 \Lambda (d\psi - \cos\theta \; d\phi)^2 \right).$$

歪んだ3次元球面 S^3

等方極限

$$ds^2 = -dt^2 + a^2\left(d\theta^2 + \sin^2\theta\; d\phi^2 + 4a^2\Lambda(d\psi - \cos\theta\; d\phi)^2\right).$$

$$2a\sqrt{\Lambda} = \sqrt{\frac{8}{\frac{mn}{\Lambda}+6}} = \sqrt{\frac{1+2v^2}{1+\frac{3}{2}v^2}}.$$

$v = 0$ とすると

$$ds^2 = -dt^2 + \frac{1}{4\Lambda}\left(d\theta^2 + \sin^2\theta\; d\phi^2 + (d\psi - \cos\theta\; d\phi)^2\right).$$

S^3

Einstein's static universe.

Axisymmetric cases

$$\frac{d^2h(\theta)}{d\theta^2} + a^2 w(\theta)h(\theta) = 0,$$

$$w(\theta) = \frac{1}{2}mn(\theta) + 3\Lambda$$

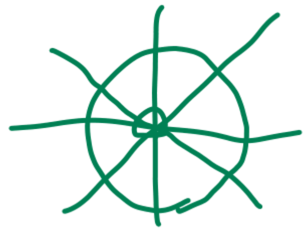
Hill 方程式

$$ds^2 = -dt^2 + a^2(d\theta^2 + h^2(\theta)d\phi^2) + (d\psi + f(\theta)d\phi)^2$$

$0 \leq \theta \leq \pi \quad \text{and} \quad 0 \leq \phi \leq 2\pi$ とすると

$$h(0) = h(\pi) = 0, \quad h'(0) = 1, \quad h'(\pi) = -1,$$

$$n'(0) = n'(\pi) = 0$$



Special cases

ダストの密度を

$$n(\theta) = n_0 - n_1 \cos(2\theta),$$ と仮定する

where n_0 and n_1 are constants.

Mathiew 方程式

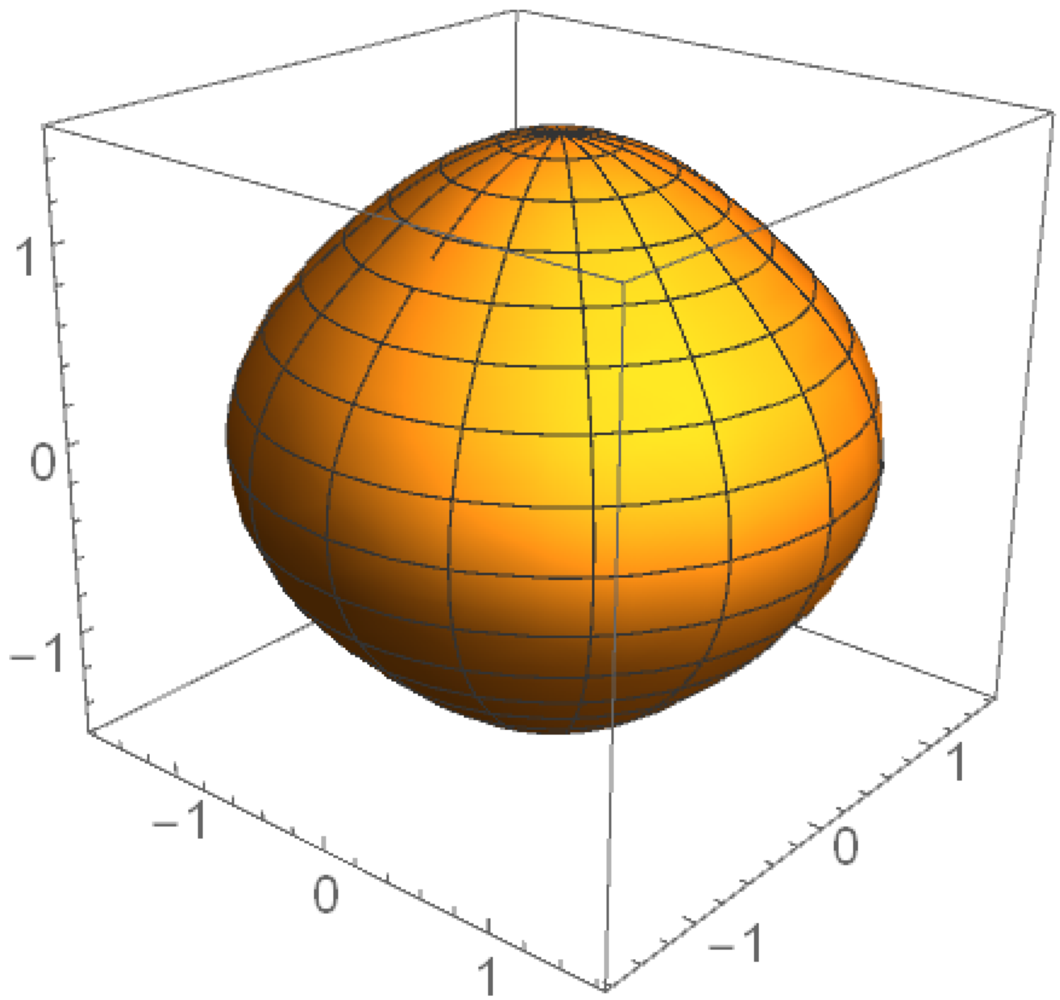
$$\frac{d^2 h}{d\theta^2} + \Big(p - 2q\cos(2\theta)\Big) h = 0,$$

$$p := \left(3\Lambda + \frac{1}{2} m n_0\right) a^2, \quad q := \frac{1}{4} m n_1 a^2.$$

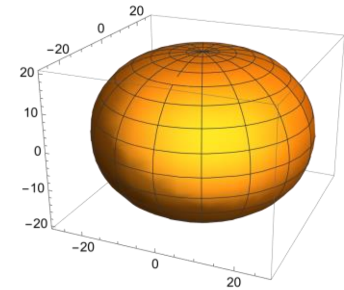
$$h(\theta) = A\ se_1(q,\theta), \qquad \frac{1}{A} = \frac{d}{d\theta} se_1(q,\theta)\Big|_{\theta=0}.$$

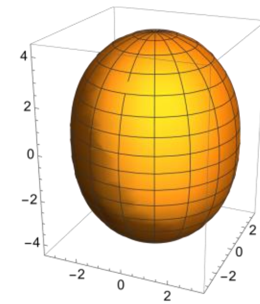
$f(\theta)$ is a primitive function of $A se_1(q,\theta)$.

2-dim. Base spaceの形



平均密度と底空間の面積





(ES) $mn(\theta) = 2\Lambda$

(i) $mn(\theta) = \Lambda(1 - \cos 2\theta)$

(ii) $mn(\theta) = \Lambda(1 + \cos 2\theta)$

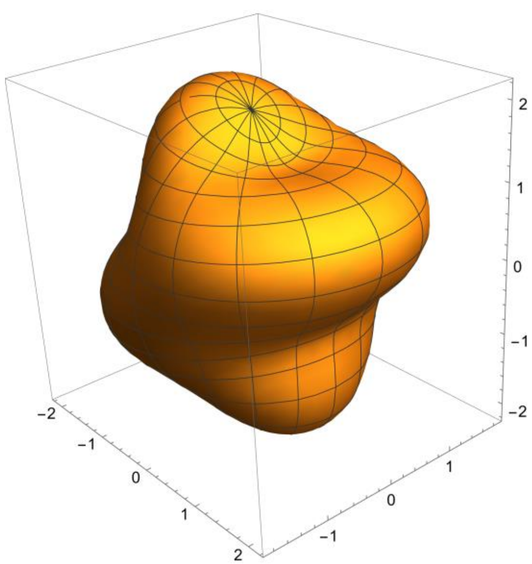
	スケール因子	Nの面積	平均密度
(ES)	$a = (1/2)\Lambda^{-1/2}$,	$A_N = \Lambda^{-1}\pi$,	$m\langle n\rangle = mn = 2\Lambda$
(i)	$a = 0.5162\Lambda^{-1/2}$,	$A_N = 1.09003\Lambda^{-1}\pi$,	$m\langle n\rangle = 1.33922\Lambda$,
(ii)	$a = 0.5545\Lambda^{-1/2}$,	$A_N = 1.19876\Lambda^{-1}\pi$,	$m\langle n\rangle = 0.673552\Lambda$.

Non-axisymmetric cases

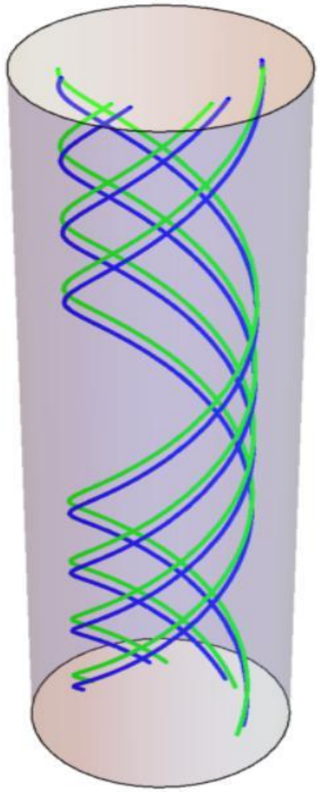
$$f(\theta, \phi) = -\cos\theta + \beta \sin^5\theta \cos\phi,$$

$$h(\theta, \phi) = \sin\theta + 5\beta \sin^4\theta \cos\theta \cos\phi,$$

$$mn(\theta, \phi) = -6\Lambda + \frac{1}{a^2}\left(2 - \frac{30\beta \sin 4\theta \cos\phi}{1 + 5\beta \sin^3\theta \cos\theta \cos\phi}\right),$$



静的解の存在



粒子の密度の濃いところは
万有引力でつぶれないか？

Raychauduhri 方程式（σ=0）

$$\frac{D}{d\tau}\theta = -\frac{1}{3}\theta^2 + \omega_{\alpha\beta}\omega^{\alpha\beta} - R_{\alpha\beta}u^\alpha u^\beta ;$$

遠心力　　曲率による引力

つりあう

$$\theta = 0$$

ここまでのまとめ

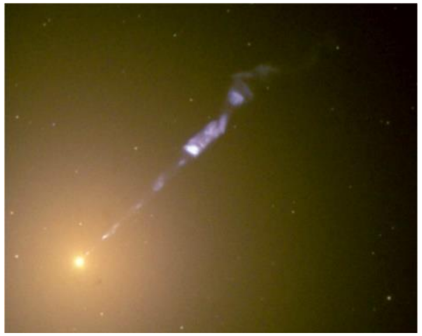
3次元Sasaki空間を用いてReebベクトル方向に運動するダスト流体を源としたEinstein方程式の静的な解を構成した.

- 空間は，非一様なS^2上のS^1バンドル.
- Einstein方程式は底空間上の線型方程式に帰着する.
- 解は底空間上の密度分布を表す任意関数を含む.

佐々木空間のねじれているがまっすぐなベクトル場を用いてEinstein重力とダスト流体の連立方程式の非一様な厳密解を構成することができる.

Dynamicalな場合？

物理的な応用・・・Jet？



M87中心からのジェット

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Basic model of classical fields

$$S = \int \sqrt{-g} d^4x \left(\frac{1}{16\pi G} R - g^{\mu\nu} (\nabla_\mu \Phi)^* (\nabla_\nu \Phi) - V(\Phi^* \Phi) \right.$$
$$\left. - \frac{1}{2} g^{\mu\nu} (\partial_\mu \Psi)(\partial_\nu \Psi) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right),$$

$$\nabla_\nu \Phi := (\partial_\nu - ieA_\nu)\Phi.$$

field equations:

$$\frac{1}{\sqrt{-g}} \nabla_\mu (\sqrt{-g} g^{\mu\nu} \nabla_\nu \Phi) - \frac{\partial V}{\partial \Phi^*} = 0,$$

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \Psi) = 0,$$

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} F^{\mu\nu}) = J^\nu,$$

$$R_{\mu\nu} = 8\pi G \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^\alpha_\alpha \right),$$

Electric current

$$J_\mu := ie\Big(\Phi^*\nabla_\mu\Phi - \Phi(\nabla_\mu\Phi)^*\Big),$$

Energy-momentum tensor

$$\begin{aligned} T_{\mu\nu} =& 2(\nabla_\mu\Phi)^*(\nabla_\nu\Phi) - g_{\mu\nu}\left(g^{\alpha\beta}(\nabla_\alpha\Phi)^*(\nabla_\beta\Phi) + V(\Phi^*\Phi)\right) \\ & + (\partial_\mu\Psi)(\partial_\nu\Psi) - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}(\partial_\alpha\Psi)(\partial_\beta\Psi) \\ & + F_{\mu\alpha}F_\nu{}^\alpha - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}. \end{aligned}$$

Metric on $\mathcal{M}^4 = \mathcal{N}^3 \times \mathbb{R}^1$

$$ds^2_{\mathcal{M}} = ds^2_{\mathcal{N}} + d\chi^2.$$

$$ds^2_{\mathcal{N}} = -(dt + f(r)d\theta)^2 + dr^2 + h^2(r)d\theta^2,$$

$$f' = 2\omega h, \quad (\omega : const.),$$

$\mathcal{N}^3$ is a quasi-Sasaki space with a Lorentzian metric

$$\sigma^0 = dt + f(r)d\theta, \quad \sigma^1 = dr, \quad \sigma^2 = hd\theta, \quad \sigma^3 = d\chi,$$

$$ds^2 = -(\sigma^0)^2 + (\sigma^1)^2 + (\sigma^2)^2 + (\sigma^3)^2.$$

$$d\sigma^0 = \frac{f'}{h}\sigma^1 \wedge \sigma^2 = 2\omega\sigma^1 \wedge \sigma^2$$

$$\sigma^0 \wedge d\sigma^0 \neq 0$$

$$ds^2_{\mathcal{N}} = -(dt + f(r)d\theta)^2 + dr^2 + h^2(r)d\theta^2,$$

$$f' = 2\omega h, \quad (\omega : const.),$$

$$\sigma^0 = dt + f(r)d\theta, \quad \sigma^1 = dr, \quad \sigma^2 = hd\theta,$$

$$d\sigma^0 = \frac{f'}{h}\sigma^1 \wedge \sigma^2 = 2\omega\sigma^1 \wedge \sigma^2$$

$$^*d\sigma^0 = 2\omega\sigma^0$$

σ^0 is a timelike Beltrami field.

We assume **VEV**

$$\Phi = v, \quad \Psi = \Psi(\chi), \quad A = B\sigma^0,$$

E.O.M.

$$\frac{1}{\sqrt{-g}}\nabla_\mu(\sqrt{-g}g^{\mu\nu}\nabla_\nu\Phi) - \frac{\partial V}{\partial\Phi^*} = 0, \quad \Longrightarrow \quad \frac{1}{2}\frac{dV(v)}{dv} = e^2B^2v,$$

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\Psi) = 0, \quad \Longrightarrow \quad \partial_\chi^2\Psi(\chi) = 0,$$

$$\Psi = k\chi + const.$$

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}F^{\mu\nu}) = J^\nu, \quad \Longrightarrow \quad \star(d\star F) = -J = -2e^2v^2A.$$

$$= -4\omega^2 A$$

$$J_\mu := ie\Big(\Phi^*\nabla_\mu\Phi - \Phi(\nabla_\mu\Phi)^*\Big),$$

$$\omega^2 = \frac{1}{2}e^2v^2$$

Einstein's equations

$$R_{\mu\nu} = 8\pi G \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^{\alpha}_{\alpha} \right)$$

Ricci curvature tensor

$$R_{ab} = 2\omega^2 \sigma^0_a \sigma^0_b + \left(2\omega^2 - \frac{h''}{h} \right) (\sigma^1_a \sigma^1_b + \sigma^2_a \sigma^2_b).$$

η Einstein

The total energy-momentum tensor

$$T_{ab} = T^{\Phi}_{ab} + T^{\Psi}_{ab} + T^{F}_{ab}$$

$$T^{\Phi}_{ab} = \left(e^2 B^2 v^2 + V(v)\right) \sigma^0_a \sigma^0_b + \left(e^2 B^2 v^2 - V(v)\right) (\sigma^1_a \sigma^1_b + \sigma^2_a \sigma^2_b + \sigma^3_a \sigma^3_b),$$

$$T^{\Psi}_{ab} = \frac{k^2}{2} (\sigma^0_a \sigma^0_b - \sigma^1_a \sigma^1_b - \sigma^2_a \sigma^2_b + \sigma^3_a \sigma^3_b),$$

$$T^{F}_{ab} = 2\omega^2 B^2 (\sigma^0_a \sigma^0_b + \sigma^1_a \sigma^1_b + \sigma^2_a \sigma^2_b - \sigma^3_a \sigma^3_b).$$

Einstein's equations reduce to

$$2\omega^2 = 2e^2B^2v^2 - V(v) + 2\omega^2B^2,$$

$$2\omega^2 - \frac{h''}{h} = V(v) + 2\omega^2B^2,$$

$$0 = V(v) + k^2 - 2\omega^2B^2.$$

Maxwell-scalar fields equations reduce to

$$B^2 = \frac{1}{2} - \frac{k^2}{2e^2v^2} = \frac{1}{3e^2v^2}(e^2v^2 + V(v)),$$

$$k^2 = \frac{1}{3}(e^2v^2 - 2V(v)),$$

$$3v\frac{dV(v)}{dv} - 2e^2v^2 - 2V(v) = 0.$$

Solutions

For wide classes of $V(\Phi^*\Phi)$.

$$h'' = 2k^2 h \qquad f' = 2\omega h, \qquad (\omega : const.),$$

For regularity, h should behave $h \to r$ as $r \to 0$,

$$h = \frac{1}{\sqrt{2}k} \sinh\sqrt{2}kr, \qquad f = \frac{2\omega}{k^2} \sinh^2 \frac{kr}{\sqrt{2}}.$$

$$ds^2 = -\left(dt + \frac{2\omega}{k^2} \sinh^2 \frac{kr}{\sqrt{2}}\, d\theta\right)^2 + dr^2 + \frac{1}{2k^2} \sinh^2 \sqrt{2}kr\, d\theta^2 + d\chi^2,$$

Gödel-type metrics:

Twisted in time

A circle of of a radius $r = const. > r_c$ on a $t = const.$ surface is a CTC.

$$\sinh^2 \frac{kr_c}{\sqrt{2}} = \left(\frac{2\omega^2}{k^2} - 1\right)^{-1}$$

existence condition $\quad \frac{1}{2} e^2 v_0^2 > V(v_0) > -e^2 v_0^2,$

$$ds^2 = -\left(dt + \frac{2\omega}{k^2}\sinh^2\frac{kr}{\sqrt{2}}\ d\theta\right)^2 + dr^2 + \frac{1}{2k^2}\sinh^2\sqrt{2}kr\ d\theta^2 + d\chi^2,$$

Gödel-type metrics:

$B \to 0.\quad k^2 = 2\omega^2$

= squashed $\mathrm{AdS}^3 \times \mathbb{R}^1$

$$ds^2 = \frac{1}{2k^2}\left(-\left(d\tau + 2\sinh^2\frac{\rho}{2}\ d\theta\right)^2 + d\rho^2 + \sinh^2\rho\ d\theta^2\right) + d\chi^2,$$

$\rho := \sqrt{2}kr$ and $\tau := \sqrt{2}kt.$

$\mathrm{AdS}^3 \times \mathbb{R}^1$

there is no CTC.

Example 1

$$V(\Phi^*\Phi) = m^2\Phi^*\Phi - V_0,$$

$$v_0^2 = \frac{V_0}{e^2 - 2m^2}, \quad B^2 = \frac{m^2}{e^2}, \quad k^2 = \frac{1}{3}V_0,$$

In the special case $e^2 = 2m^2$ and $V_0 = 0$,

$$B^2 = \frac{1}{2}, \quad k^2 = 0,$$

$$ds^2 = -\left(dt + \omega r^2 d\theta\right)^2 + dr^2 + r^2 d\theta^2 + d\chi^2,$$

Example 2

$$V(\Phi) = \frac{\lambda}{2}\left(\Phi^*\Phi - \eta^2\right)^2 - e^2\eta^2$$

$$\text{(i)} \quad v_0^2 = \eta^2, \qquad \text{(ii)} \quad v_0^2 = \frac{1}{5\lambda}(2e^2 - \lambda\eta^2).$$

$$B^2 = 0, \quad k^2 = e^2\eta^2,$$

$$B^2 = \frac{2}{5e^2}(e^2 - 3\lambda\eta^2), \quad k^2 = \frac{1}{25\lambda}\left(2e^4 + 23e^2\lambda\eta^2 - 12\lambda^2\eta^4\right)$$

$$\mathrm{AdS}^3 \times \mathbb{R}^1$$

3-dimensional Lorentzian Sasaki space を用いて
古典場の理論の時空解を構成した.

For a model, there exist solutions of 3 types

Gödel-type metrics:

$$\mathrm{AdS}^3 \times \mathbb{R}^1$$

$$\mathrm{AdS}^4$$

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Current-flux compactification

高次元時空モデルにおいて，余剰次元のコンパクト化に一様磁場やモノポール磁場を用いるものがある．
接触1形式（Beltrami 場）を用いて，ねじれた磁場によるコンパクト化を示す．

We consider a metric on a fiber bundle

$$
\begin{aligned}
ds^2 =& ds_M^2 + (\sigma^1)^2 + (\sigma^2)^2 + (\sigma^3)^2 + (\sigma^4)^2 + (\sigma^5)^2 \\
=& ds_M^2 + \left(d\theta_1^2 + h_1(\theta_1, \phi_1)^2 \ d\phi_1^2\right) + \left(d\theta_2^2 + h_2(\theta_2, \phi_2)^2 \ d\phi_2^2\right) \\
& + \left(d\psi + p \ f_1(\theta_1, \phi_1) \ d\phi_1 + q \ f_2(\theta_2, \phi_2) \ d\phi_2\right)^2 ,
\end{aligned}
$$

where p, q are constants, $f_i(\theta_i, \phi_i)$ are arbitrary functions, and the functions $h_i(\theta_i, \phi_i)$ are

$$
h_1(\theta_1, \phi_1) := \frac{\partial f_1(\theta_1, \phi_1)}{\partial \theta_1}, \quad h_2(\theta_2, \phi_2) := \frac{\partial f_2(\theta_2, \phi_2)}{\partial \theta_2}.
$$

The metric admits Killing vectors

$$
\xi_{(t)} = \partial_t, \quad \xi_{(\psi)} = \partial_\psi,
$$

$$\eta = \sigma^5 = d\psi + p\ f_1(\theta_1, \phi_1)\ d\phi_1 + q\ f_2(\theta_2, \phi_2)\ d\phi_2$$

$$\begin{aligned} d\eta = d\sigma^5 &= p\partial_{\theta_1} f_1(\theta_1, \phi_1) d\theta_1 \wedge d\phi_1 + q\partial_{\theta_2} f_2(\theta_2, \phi_2) d\theta_2 \wedge d\phi_2 \\ &= p d\theta_1 \wedge h_1(\theta_1, \phi_1) d\phi_1 + q d\theta_2 \wedge h_2(\theta_2, \phi_2) d\phi_2 \\ &= p\sigma^1 \wedge \sigma^2 + q\sigma^3 \wedge \sigma^4. \end{aligned}$$

$$\eta \wedge d\eta \wedge d\eta = \sigma^5 \wedge d\sigma^5 \wedge d\sigma^5 = pq\ \sigma^5 \wedge \sigma^1 \wedge \sigma^2 \wedge \sigma^3 \wedge \sigma^4 \neq 0$$

$\eta = \sigma^5$ is a contact form.

$$\eta(\partial_\psi) = d\psi(\partial_\psi) = 1,$$

$$d\eta(\partial_\psi,\ ) = 0$$

$\xi_{(\psi)} = \partial_\psi$ is the '*Reeb vector*' associated to η.

$$\eta = \sigma^5 = d\psi + p\ f_1(\theta_1, \phi_1)\ d\phi_1 + q\ f_2(\theta_2, \phi_2)\ d\phi_2$$

$$d\eta = p\sigma^1 \wedge \sigma^2 + q\sigma^3 \wedge \sigma^4$$

$$d\eta \wedge d\eta = pq\sigma^1 \wedge \sigma^2 \wedge \sigma^3 \wedge \sigma^4$$

$$^*(d\eta \wedge d\eta) = pq\ \sigma^5 = pq\ \eta$$

η : "Beltrami field" in 5 dimension

$$g_{ab} = \sigma^1\sigma^1 + \sigma^2\sigma^2 + \sigma^3\sigma^3 + \sigma^4\sigma^4 + \sigma^5\sigma^5$$

$$g^{ab} = e_1e_1 + e_2e_2 + e_3e_3 + e_4e_4 + e_5e_5$$

$$\sigma^i(e_j) = \delta^i{}_j$$

$$\phi^2(X) = -X + \eta(X)\xi$$

$$\begin{aligned}
\phi^a{}_b &= g^{ac}(p'\ \sigma^1 \wedge \sigma^2 + q'\ \sigma^3 \wedge \sigma^4)_{cb} \\
&= p'\ (e_1e_1 + e_2e_2)^{ac}(\sigma^1 \wedge \sigma^2)_{cb} + q'(e_3e_3 + e_4e_4)^{ac}\ (\sigma^3 \wedge \sigma^4)_{cb} \\
&= \frac{1}{2}p'\ (e_1\sigma^2 - e_2\sigma^1)^a{}_b + \frac{1}{2}q'(e_3\sigma^4 - e_4\sigma^3)^a{}_b
\end{aligned}$$

$$\phi^a{}_b = \frac{1}{2}p' \, (e_1\sigma^2 - e_2\sigma^1)^a{}_b + \frac{1}{2}q'(e_3\sigma^4 - e_4\sigma^3)^a{}_b$$

$$\phi^2(X) = -X + \eta(X)\xi$$

$$\begin{aligned}\phi^2 &= \phi^a{}_c\phi^c{}_b \\ &= \left(\frac{1}{2}p' \, (e_1\sigma^2 - e_2\sigma^1)^a{}_c + \frac{1}{2}q'(e_3\sigma^4 - e_4\sigma^3)^a{}_c\right)\left(\frac{1}{2}p' \, (e_1\sigma^2 - e_2\sigma^1)^c{}_b + \frac{1}{2}q'(e_3\sigma^4 - e_4\sigma^3)^c{}_b\right) \\ &= \frac{1}{4}p'^2 \, (-e_1\sigma^1 - e_2\sigma^2)^a{}_b + \frac{1}{4}q'^2(-e_3\sigma^3 - e_4\sigma^4)^a{}_b\end{aligned}$$

$$p'^2 = q'^2 = 4$$

$$\begin{aligned}\phi^2 &= \left(-e_1\sigma^1 - e_2\sigma^2 - e_3\sigma^3 - e_4\sigma^4 - e_5\sigma^5\right)^a{}_b + (e_5\sigma^5)^a{}_b \\ &= -\delta^a{}_b + \xi^a\eta{}_b\end{aligned}$$

Almost contact manifold

$$\begin{aligned} g(\phi X, \phi Y) &= g_{ab}\phi^a{}_c X^c \phi^b{}_d Y^d \\ &= g_{ab}\phi^a{}_c \phi^b{}_d X^c Y^d \\ &= (g_{ab} - \eta_a \eta_b) X^a Y^b \\ &= g(X, Y) - \eta(X)\eta(Y) \end{aligned}$$

almost contact metric manifold

$$ds^2_{\mathcal{M}^5} = \left(d\theta_1^2 + h_1(\theta_1,\phi_1)^2\ d\phi_1^2\right) + \left(d\theta_2^2 + h_2(\theta_2,\phi_2)^2\ d\phi_2^2\right)$$
$$+ \left(d\psi + p\ f_1(\theta_1,\phi_1)\ d\phi_1 + q\ f_2(\theta_2,\phi_2)\ d\phi_2\right)^2,$$

If $p = p' = \pm 2, q = q' = \pm 2,$

$$d\eta_{ab} = 2(\sigma^1 \wedge \sigma^2 + \sigma^3 \wedge \sigma^4)_{ab} = g_{ac}\phi^c{}_b$$

$(\mathcal{M}^5, \eta, \xi, \phi, g_{\mathcal{M}^5})$ is a contact metric manifold

MAXWELL FIELD

$$dF = 0, \quad *d*F = -j.$$

$$F = dA,$$

$$A = d\psi + \bar{p}\ f_1(\theta_1, \phi_1)\ d\phi_1 + \bar{q}\ f_2(\theta_2, \phi_2)\ d\phi_2$$

$$F = dA = \bar{p}\ \sigma^1 \wedge \sigma^2 + \bar{q}\ \sigma^3 \wedge \sigma^4$$

$$F_{12} = -F_{21} = \bar{p}, \quad F_{34} = -F_{43} = \bar{q}$$

$$
\begin{aligned}
*d*F &= *d*(\bar{p}\ \sigma^1 \wedge \sigma^2 + \bar{q}\ \sigma^3 \wedge \sigma^4) \\
&= -\frac{1}{2} * d(\bar{p}\ \sigma^3 \wedge \sigma^4 \wedge \sigma^5 \wedge \sigma^0 + \bar{q}\ \sigma^5 \wedge \sigma^0 \wedge \sigma^1 \wedge \sigma^2) \\
&= -\frac{1}{2} * (\bar{p}\ \sigma^3 \wedge \sigma^4 \wedge d\sigma^5 \wedge \sigma^0 + \bar{q}\ d\sigma^5 \wedge \sigma^0 \wedge \sigma^1 \wedge \sigma^2) \\
&= -\frac{1}{2} * (\bar{p}p\ \sigma^3 \wedge \sigma^4 \wedge \sigma^1 \wedge \sigma^2 \wedge \sigma^0 + \bar{q}q\ \sigma^3 \wedge \sigma^4 \wedge \sigma^0 \wedge \sigma^1 \wedge \sigma^2) \\
&= -\frac{1}{2} * (\bar{p}p\ \sigma^0 \wedge \sigma^1 \wedge \sigma^2 \wedge \sigma^3 \wedge \sigma^4 + \bar{q}q\ \sigma^0 \wedge \sigma^1 \wedge \sigma^2 \wedge \sigma^3 \wedge \sigma^4) \\
&= \frac{1}{2}(\bar{p}p + \bar{q}q)\ \ \eta\ = -j.
\end{aligned}
$$

If $\bar{p}p + \bar{q}q = 0$, $F = dA$ is a vacuum solution to Maxwell's equations.

Ricci curvature

$$R_{00} = 0,$$

$$R_{11} = R_{22} = -\frac{p^2}{2} - \frac{\partial^2_{\theta_1} h_1(\theta_1, \phi_1)}{h_1(\theta_1, \phi_1)},$$

$$R_{33} = R_{44} = -\frac{q^2}{2} - \frac{\partial^2_{\theta_2} h_2(\theta_2, \phi_2)}{h_2(\theta_2, \phi_2)},$$

$$R_{55} = \frac{p^2 + q^2}{2}.$$

$$R_{ab} = \beta_1(\theta_1, \phi_1) g^1_{ab}(\theta_1, \phi_1) + \beta_2(\theta_2, \phi_2) g^2_{ab}(\theta_2, \phi_2) + \alpha\ \eta_a \eta_b$$

$$R_{ab} = \beta_1(\theta_1, \phi_1) g^1_{ab}(\theta_1, \phi_1) + \beta_2(\theta_2, \phi_2) g^2_{ab}(\theta_2, \phi_2) + \alpha\ \eta_a \eta_b$$

If we require $\beta_1(\theta_1, \phi_1) = \beta_2(\theta_2, \phi_2) = \beta$,

$$R_{ab} = \beta\big(g^1_{ab}(\theta_1, \phi_1) + g^2_{ab}(\theta_2, \phi_2)\big) + \alpha\ \eta_a \eta_b$$

$$= \beta\ g_{ab} + \gamma\ \eta_a \eta_b$$

η Einstein $\quad \beta,\ \gamma$ are constants.

$$ds^2 = -dt^2 + ds^2_{\mathcal{M}^5},$$

Einstein' equations in 6-dim.

$$R_{ab} - \frac{1}{2}g_{ab}R + \Lambda g_{ab} = T_{ab}$$

trace

$$g^{ab}R_{ab} - \frac{1}{2}g^{ab}g_{ab}R + \Lambda g^{ab}g_{ab} = g^{ab}T_{ab}$$

$$R - \frac{6}{2}R + 6\Lambda = -2R + 6\Lambda = g^{ab}T_{ab}$$

$$R = 3\Lambda - \frac{1}{2}T$$

Another form of Einstein's equations

$$R_{ab} = T_{ab} + \frac{1}{2}g_{ab}R$$

$$= T_{ab} - \frac{1}{4}g_{ab}T + \frac{3}{2}g_{ab}\Lambda$$

Energy-momentum tensor of the Maxwell field

$$F = dA = \bar{p}\ \sigma^1 \wedge \sigma^2 + \bar{q}\ \sigma^3 \wedge \sigma^4$$

$$T^{EM}_{\mu\nu} = F_{\mu\sigma} F_{\nu\lambda} g^{\sigma\lambda} - \frac{1}{4} g_{\mu\nu} F_{\alpha\lambda} F_{\beta\sigma} g^{\alpha\beta} g^{\lambda\sigma}$$

$$F^2 := F_{\alpha\lambda} F_{\beta\sigma} g^{\alpha\beta} g^{\lambda\sigma} = 2F_{12}F_{12}g^{11}g^{22} + 2F_{34}F_{34}g^{33}g^{44} = 2(\bar{p}^2 + \bar{q}^2)$$

$$T^{EM}_{11} = T^{EM}_{22} = F_{12}F_{12}g^{22} - \frac{1}{4} g_{11} F^2 = \bar{p}^2 - \frac{1}{2}(\bar{p}^2 + \bar{q}^2)$$

$$= \frac{1}{2}(\bar{p}^2 - \bar{q}^2)$$

$$T^{EM}_{33} = T^{EM}_{44} = F_{34}F_{34}g^{44} - \frac{1}{4} g_{33} F^2 = \bar{q}^2 - \frac{1}{2}(\bar{p}^2 + \bar{q}^2)$$

$$= \frac{1}{2}(\bar{q}^2 - \bar{p}^2)$$

$$T^{EM}_{55} = -\frac{1}{4} g_{55} F^2 = -\frac{1}{2}(\bar{p}^2 + \bar{q}^2)$$

$$T^{EM}_{00} = -\frac{1}{4} g_{00} F^2 = \frac{1}{2}(\bar{p}^2 + \bar{q}^2)$$

$$\mathrm{tr} T = F^2 - \frac{6}{4} F^2 = -\frac{1}{2} F^2 = -\bar{p}^2 + \bar{q}^2$$

Dust currents

$$J^{\mu} = (en_+ u_+^{\mu} + (-e)n_- u_-^{\mu}),$$

$$u_{\pm}^{\mu} = (u_{\pm}^0, 0, 0, 0, 0, u_{\pm}^5)$$

$$u_{\pm}^b = (u_{\pm}^0 e_0 + u_{\pm}^5 e_5)^b = (u_{\pm}^0 \partial_t + u_{\pm}^5 \partial_{\psi})^b$$

$$u_{\pm}^b \nabla_b u_{\pm}^a = 0.$$

$$J^0 = (en_+ u_+^0 - en_- u_-^0) = 0,$$

$$J^1 = J^2 = J^3 = J^4 = 0,$$

$$J^5 = (en_+ u_+^5 + (-e)n_- u_-^5).$$

$$\frac{1}{2}n = n_+ = n_-, \quad u^5_+ = -u^5_-, \quad u^0_+ = u^0_-,$$

Counter flow

$$J^5 = enu^5_+, \quad J^0 = 0, \quad J^1 = J^2 = J^3 = J^4 = 0,$$

$$J^a = enu^5_+ \partial_\psi,$$

$$J_a = enu^+_5 \sigma^5.$$

Maxwell equations,

$$*d*F = \frac{1}{2}(\bar{p}p + \bar{q}q)\ \sigma^5 \ = -J$$

$$\frac{1}{2}(\bar{p}p + \bar{q}q)\ \sigma^5 = -enu^+_5 \sigma^5$$

Energy-momentum tensor of the dust currents

$$T_J{}^{\mu\nu} = m_+ n_+ u_+^\mu u_+^\nu + m_- n_- u_-^\mu u_-^\nu,$$

$$T_J{}^{00} = m_+ n_+ u_+^0 u_+^0 + m_- n_- u_-^0 u_-^0,$$

$$T_J{}^{05} = m_+ n_+ u_+^0 u_+^5 + m_- n_- u_-^0 u_-^5 = 0,$$

$$T_J{}^{55} = m_+ n_+ u_+^5 u_+^5 + m_- n_- u_-^5 u_-^5,$$

others are vanishing

$$n_+ u_+^0 = n_- u_-^0,$$
$$m_+ u_+^5 = -m_- u_-^5.$$

$$\frac{1}{2} n = n_+ = n_-, \; m = m_+ = m_-,$$

$$T_J{}^{00} = \left(\frac{m_+ + m_-}{n_+} \right) (n_+ u_+^0)^2 = nm (u_+^0)^2 = nm \frac{1}{1 - v^2}$$
$$T_J{}^{55} = \left(\frac{m_+ + m_-}{n_+} \right) (n_+ u_+^5)^2 = nm (u_+^5)^2 = nm \frac{v^2}{1 - v^2}$$

Einstein’s equations

$$R_{ab} = S^{EM}_{ab} + S^{J}_{ab} + \frac{3}{2}\Lambda g_{ab} \qquad S_{ab} = T_{ab} - \frac{1}{4}(trT\ )g_{ab}$$

$$R_{00} = 0 = \frac{1}{4}3(\bar{p}^2 + \bar{q}^2) + \frac{mn}{4}\frac{3+v^2}{1-v^2} - \frac{3}{2}\Lambda,$$

$$R_{11} = -\frac{p^2}{2} - \frac{h_1''(\theta_1,\phi_1)}{h_1(\theta_1,\phi_1)} = \frac{1}{4}(3\bar{p}^2 - \bar{q}^2) + \frac{mn}{4} + \frac{3}{2}\Lambda,$$

$$R_{33} = -\frac{q^2}{2} - \frac{h_2''(\theta_2,\phi_2)}{h_2(\theta_2,\phi_2)} = \frac{1}{4}(3\bar{q}^2 - \bar{p}^2) + \frac{mn}{4} + \frac{3}{2}\Lambda,$$

$$R_{55} = \frac{p^2+q^2}{2} = -\frac{1}{4}(\bar{p}^2 + \bar{q}^2) + \frac{mn}{4}\frac{1+3v^2}{1-v^2} + \frac{3}{2}\Lambda.$$

$$-\frac{h_1''(\theta_1,\phi_1)}{h_1(\theta_1,\phi_1)} = \lambda_+$$

$$-\frac{h_2''(\theta_2,\phi_2)}{h_2(\theta_2,\phi_2)} = \lambda_-$$

$$\lambda_+ := \frac{p^2}{2} + \frac{1}{4}(3\bar{p}^2 - \bar{q}^2) + \frac{mn}{4} + \frac{3}{2}\Lambda,$$

$$\lambda_- := \frac{q^2}{2} + \frac{1}{4}(3\bar{q}^2 - \bar{p}^2) + \frac{mn}{4} + \frac{3}{2}\Lambda,$$

$$v^2 = \frac{p\bar{p} + q\tilde{q}}{p\bar{p} + q\bar{q} - 2en}$$

$$p^2 + q^2 = \bar{p}^2 + \bar{q}^2 + 2mn\frac{1+v^2}{1-v^2},$$

$$2(p^2 + q^2) = -(\bar{p}^2 + \bar{q}^2) + mn\frac{1+3v^2}{1-v^2} + 6\Lambda.$$

$$ds^2_{\mathcal{M}^5} = \left(d\theta_1^2 + \sin^2(\sqrt{\lambda_+}\ \theta_1)\ d\phi_1^2\right) + \left(d\theta_2^2 + \sin^2(\sqrt{\lambda_-}\ \theta_2)\ d\phi_2^2\right)$$
$$+ \left(d\psi + p\ \cos(\sqrt{\lambda_+}\ \theta_1)\ d\phi_1 + q\ \cos(\sqrt{\lambda_-}\ \theta_2)\ d\phi_2\right)^2,$$

$$\tilde{\theta}_1 := \sqrt{\lambda_+}\ \theta_1, \quad \tilde{\theta}_2 := \sqrt{\lambda_-}\ \theta_2,$$
$$\tilde{\phi}_1 := \sqrt{\lambda_+}\ \phi_1, \quad \tilde{\phi}_2 := \sqrt{\lambda_-}\ \phi_2,$$
$$\tilde{p} := \frac{p}{\lambda_+}, \quad \tilde{q} := \frac{q}{\lambda_-}.$$

$T^{\tilde{p},\tilde{q}}$

$$ds^2_{\mathcal{M}^5} = \frac{1}{\lambda_+}\left(d\tilde{\theta}_1^2 + \sin^2\tilde{\theta}_1\ d\tilde{\phi}_1^2\right) + \frac{1}{\lambda_-}\left(d\tilde{\theta}_2^2 + \sin^2\tilde{\theta}_2\ d\tilde{\phi}_2^2\right)$$
$$+ \left(d\psi + \tilde{p}\ \cos\tilde{\theta}_1\ d\tilde{\phi}_1 + \tilde{q}\ \cos\tilde{\theta}_2\ d\tilde{\phi}_2\right)^2,$$

ねじれた磁場であるBeltrami磁場をbackground として
もつような高次元時空解が得られた.

ありがとうございました.