Alfvén waves in a homogeneous Beltrami magnetic field

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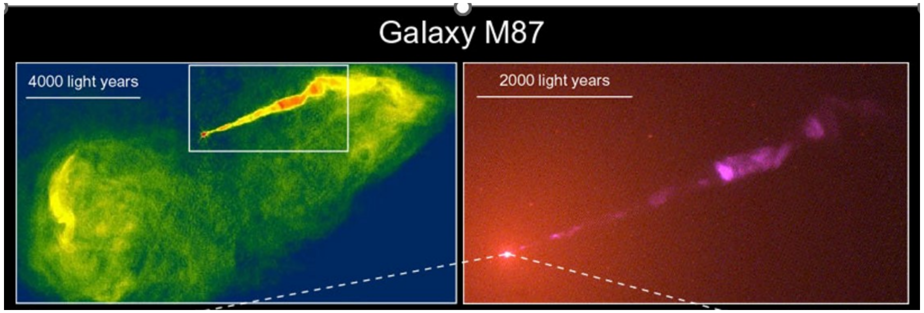
Introduction



- As was shown by the yesterday's talk by Koide-san and Noda-san, there are many energetic astrophysical phenomena.
- Jets would be collimated by magnetic fields around black holes.
- Energy of the jets would be carried by Alfvén waves.

Alfvén waves are transverse waves in plasma propagating along a magnetic field.

Magnetic fields around Kerr black holes would be twisted.

Alfvén waves propagating along a homogeneous magnetic field

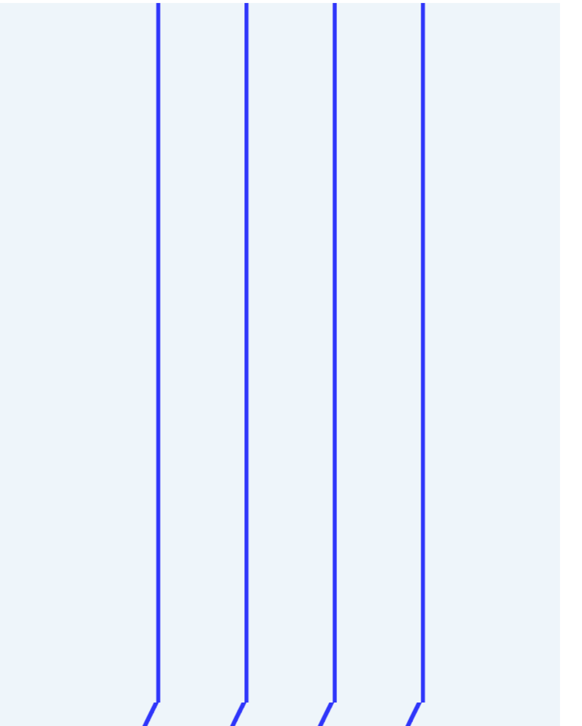
- Wave front is perpendicular to the magnetic field.
- Poynting vector is along the magnetic field.

We investigate Alfvén waves propagating on a twisted magnetic fields.

According to Frobenius' theorem, there is no surface orthogonal to a twisted vector field.

Naïve questions:

- What is a wave front?
- What direction of Poynting vector?



We investigate Alfvén waves propagating on a Beltrami magnetic fields.

A Beltrami vector field:

$$\mathrm{rot}\ \boldsymbol{B} = \lambda \boldsymbol{B}$$

$$\boldsymbol{B} \cdot \mathrm{rot}\ \boldsymbol{B} = \lambda |\boldsymbol{B}|^2 \neq 0$$

Twisted vector field

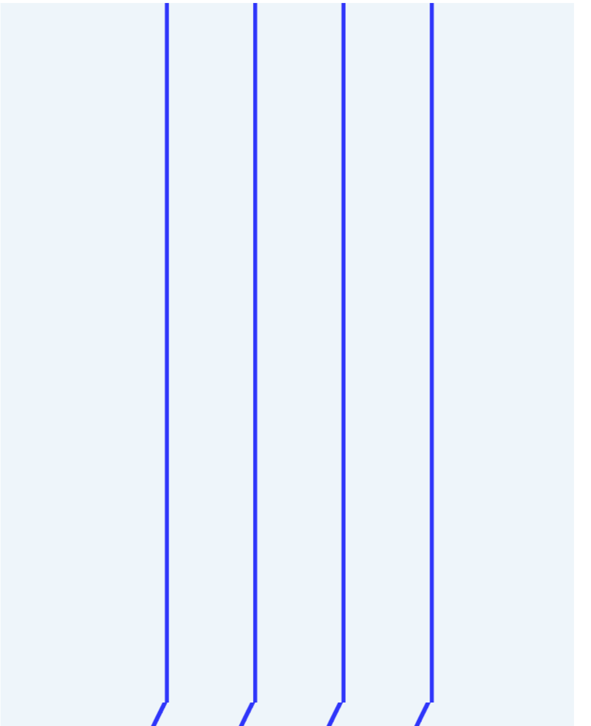
S. M. Mahajan, V. Krishan. Exact solution of the incompressible hall magnetohydrodynamics. Mon Not Roy Astron Soc 2005;359:L27-29

Z. Yoshida . Nonlinear Alfvén/Beltrami waves-An integrable structure built around the Casimir. Commum Nonlinear Sci Numer Simulat 2011.

Plan of this talk

1. Alfvén waves on a homogeneous magnetic field without twist
2. Alfvén waves on a homogeneous Beltrami magnetic field

1. Alfvén waves on a homogeneous magnetic field without twist



4-dimensional notations

$$\mu, \nu = 0, 1, 2, 3,$$
$$i, j = 1, 2, 3$$

4-vector potential $A_\mu = (-\phi, A_i), \quad (c = 1)$

Field strength $F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu$

$$E_i := F_{it}, \quad B_i := -{}^*F_{it}, \quad {}^*F_{\mu\nu} := \frac{1}{2}\epsilon_{\mu\nu}{}^{\lambda\rho}F_{\lambda\rho}$$

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}$$

4-velocity $U^\mu := \dfrac{dx^\mu}{d\tau}, \quad d\tau^2 := -ds^2 = dt^2 - \delta_{ij}dx^i dx^j$

$$U^0 = \frac{dx^0}{d\tau} = \frac{1}{\sqrt{1 - v^2}}, \quad U^i = \frac{dx^i}{d\tau} = \frac{v^i}{\sqrt{1 - v^2}}$$

4-current $J^\mu := \rho_e U^\mu$

Basic equations

Maxwell equations

$$\nabla_\mu F^{\mu\nu} = -J^\nu$$

$$\nabla_\mu F_{\nu\lambda} + \nabla_\nu F_{\lambda\mu} + \nabla_\lambda F_{\mu\nu} = 0$$

Euler equation

$$\rho U^\mu \nabla_\mu U^\nu = F^\nu{}_\lambda J^\lambda - \nabla^\nu P$$

Ideal MHD condition

$$F_{\mu\nu} U^\nu = 0$$

Continuity equation

$$\nabla_\mu (\rho U^\mu) = 0$$

Current conservation

$$\nabla_\mu J^\mu = 0$$

Background solution with untwisted magnetic field

$$U = U_{(0)} + u, \quad J = J_{(0)} + j$$

$$F = F^{(0)} + f$$

Background

$$U_{(0)} = \partial_t, \quad J_{(0)} = 0$$

$$F^{(0)} = B\ dx \wedge dy, \quad F^{(0)}_{xy} = B^z = B = const.$$

- Plasma is at rest,
- No electric current,
- Homogeneous magnetic field without twist

Background configurations solve the basic equations.

Perturbations

Assumptions

$$u^t = 0, \quad u^z = 0,$$

$$j^t = 0, \quad j^z = 0,$$

$$f_{xy} = 0, \quad f_{tz} = 0,$$

Nonvanishing components

$$u^x, \quad u^y, \quad j^x, \quad j^y,$$

$$f_{tx}, \quad f_{ty}, \quad f_{zx}, \quad f_{zy}$$

$$\boldsymbol{e} := \begin{pmatrix} e_x \\ e_y \end{pmatrix} = \begin{pmatrix} f_{xt} \\ f_{yt} \end{pmatrix}$$

$$\boldsymbol{b} := \begin{pmatrix} b_x \\ b_y \end{pmatrix} = \begin{pmatrix} f_{yz} \\ f_{zx} \end{pmatrix}$$

These variables are assumed to be dependent on t and z.

$$\boldsymbol{u} \cdot \boldsymbol{B} = 0, \quad \boldsymbol{j} \cdot \boldsymbol{B} = 0,$$

$$\boldsymbol{e} \cdot \boldsymbol{B} = 0, \quad \boldsymbol{b} \cdot \boldsymbol{B} = 0,$$

Euler equations for perturbations

$$\rho \partial_t u^\nu = g^{\nu\sigma}(F^{(0)}_{\sigma\lambda} J^\lambda_{(0)} + F^{(0)}_{\sigma\lambda} j^\lambda + f_{\sigma\lambda} J^\lambda_{(0)})$$

$$= g^{\nu\sigma} F^{(0)}_{\sigma\lambda} j^\lambda$$

$\nu = x, y$

$$\rho \partial_t u^x = F^x_{(0)y} j^y = B j^y$$

$$\rho \partial_t u^y = F^y_{(0)x} j^x = -B j^x$$

Euler eqution

$$j^x = -\frac{\rho}{B}\partial_t u^y, \quad j^y = \frac{\rho}{B}\partial_t u^x;$$

$$\boldsymbol{j} = -\frac{\rho}{B}\partial_t(\epsilon \cdot \boldsymbol{u})$$

$$\epsilon := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \boldsymbol{u} := \begin{pmatrix} u^x \\ u^y \end{pmatrix}$$

Ideal MHD conditions

$$F_{\mu\nu}U^{\nu} = F^{(0)}_{\mu\nu}u^{\nu} + f_{\mu t}U^{t}_{(0)} = 0$$

$$\mu = x, y \qquad F^{(0)}_{xy}u^{y} + f_{xt} = 0$$

$$F^{(0)}_{yx}u^{x} + f_{yt} = 0$$

Ideal MHD condition

$$u^{x} = -\frac{1}{B}f_{ty}, \quad u^{y} = \frac{1}{B}f_{tx}$$

$$\boldsymbol{u} = \frac{1}{B}\epsilon \cdot \boldsymbol{e}$$

$$\boldsymbol{e} := \begin{pmatrix} e_x \\ e_y \end{pmatrix} = \begin{pmatrix} f_{xt} \\ f_{yt} \end{pmatrix}$$

Faraday's law

Bianchi identity

$$\partial_\mu f_{\nu\lambda} + \partial_\nu f_{\lambda\mu} + \partial_\lambda f_{\mu\nu} = 0$$

$$(\mu, \nu, \lambda) = (t, z, x), (t, z, y)$$

$$\partial_t f_{xz} + \partial_z f_{tx} = 0$$

$$\partial_t f_{yz} + \partial_z f_{ty} = 0$$

Faraday's law

$$\partial_t(\epsilon \cdot \boldsymbol{b}) = \partial_z \boldsymbol{e}$$

Ampére-Maxwell's law

Maxwell equation

$$j^{\nu} = -\partial_{\mu} f^{\mu\nu};$$

$\nu = x, y$

$$j^{x} = -\partial_{t} f^{tx} - \partial_{z} f^{zx}$$

$$j^{y} = -\partial_{t} f^{ty} - \partial_{z} f^{zy}$$

Ampere-Maxwell law

$$\boldsymbol{j} = -\partial_{t}\boldsymbol{e} - \partial_{z}(\epsilon \cdot \boldsymbol{b})$$

Equations for perturbations

Euler equations: $\boldsymbol{j} = -\frac{\rho}{B}\partial_t(\epsilon \cdot \boldsymbol{u})$

MHD conditions: $\boldsymbol{u} = \frac{1}{B}\epsilon \cdot \boldsymbol{e}$

Faraday's law: $\partial_t(\epsilon \cdot \boldsymbol{b}) = \partial_z \boldsymbol{e}$

Ampere-Maxwell's law: $\boldsymbol{j} = -\partial_t \boldsymbol{e} - \partial_z(\epsilon \cdot \boldsymbol{b})$

Erasing j and u, we have wave equations.

Ampere-Maxwell's law Euler equation

$$-\partial_t \boldsymbol{e} - \partial_z(\epsilon \cdot \boldsymbol{b}) = \boldsymbol{j} = -\frac{\rho}{B}\partial_t(\epsilon \cdot \boldsymbol{u})$$

MHD $\boldsymbol{u} = \frac{1}{B}\epsilon \cdot \boldsymbol{e}$ →

$$= -\frac{\rho}{B}\partial_t(\epsilon \cdot \frac{1}{B}\epsilon \cdot \boldsymbol{e})$$

$$= \frac{\rho}{B^2}\partial_t \boldsymbol{e}$$

Apply ∂_t

$$-\partial_t^2 \mathbf{e} + \partial_z \partial_t(\epsilon \cdot \boldsymbol{b}) = \frac{\rho}{B^2}\partial_t^2 \mathbf{e}$$

Faraday's law $\partial_t(\epsilon \cdot \boldsymbol{b}) = \partial_z \boldsymbol{e}$

$$-\partial_t^2 \mathbf{e} + \partial_z^2 \mathbf{e} = \frac{\rho}{B^2}\partial_t^2 \mathbf{e}$$

$$\boxed{-\frac{1}{c_a^2}\partial_t^2 \boldsymbol{e} + \partial_z^2 \boldsymbol{e} = 0}$$

$$\frac{1}{c_a^2} := 1 + \frac{\rho}{B^2}$$

Wave equations for Alfvén waves

$$-\frac{1}{c_a^2}\partial_t^2 \boldsymbol{e} + \partial_z^2 \boldsymbol{e} = 0$$

$$-\frac{1}{c_a^2}\partial_t^2 \boldsymbol{b} + \partial_z^2 \boldsymbol{b} = 0$$

$$\boldsymbol{u} = \frac{1}{B}\epsilon \cdot \boldsymbol{e}$$

$$-\frac{1}{c_a^2}\partial_t^2 \boldsymbol{u} + \partial_z^2 \boldsymbol{u} = 0$$

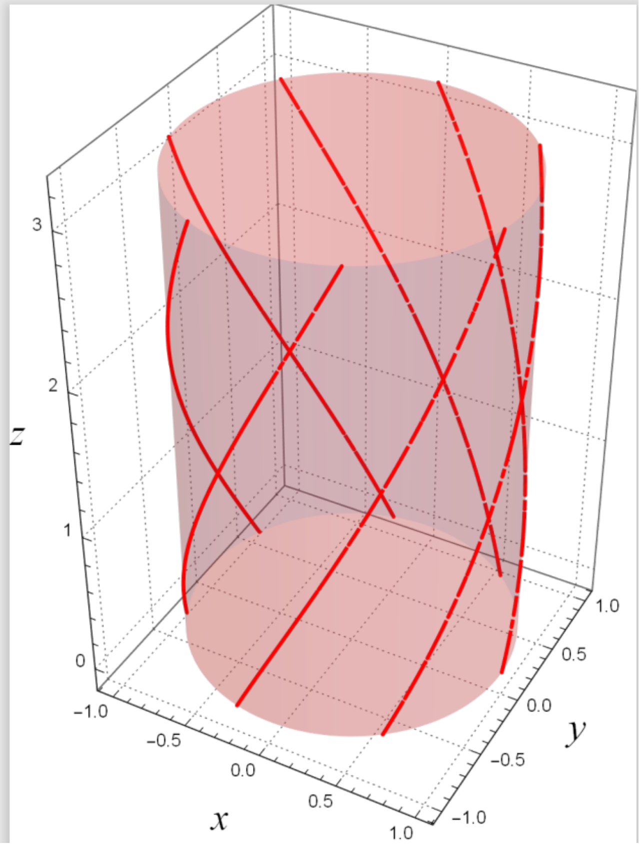
$$\frac{1}{c_a^2} := 1 + \frac{\rho}{B^2} \quad : \text{Alfvén velocity}$$

Dispersion relation

$$\frac{1}{c_a^2}\omega^2 = k^2$$

$$\boldsymbol{b} = \frac{k}{\omega}\epsilon \cdot \boldsymbol{e} = \frac{k}{\omega}B\boldsymbol{u},$$

Alfvén waves on a homogeneous Beltrami magnetic field



Twisted fiber-bundle

Metric of $R^1 \times S^3$ (time$\times$ 3-dim.sphere) as a spacetime

$$ds^2 = -dt^2 + l^2\left((d\theta^2 + \sin^2\theta d\phi^2) + (d\psi - \cos\theta d\phi)^2\right)$$

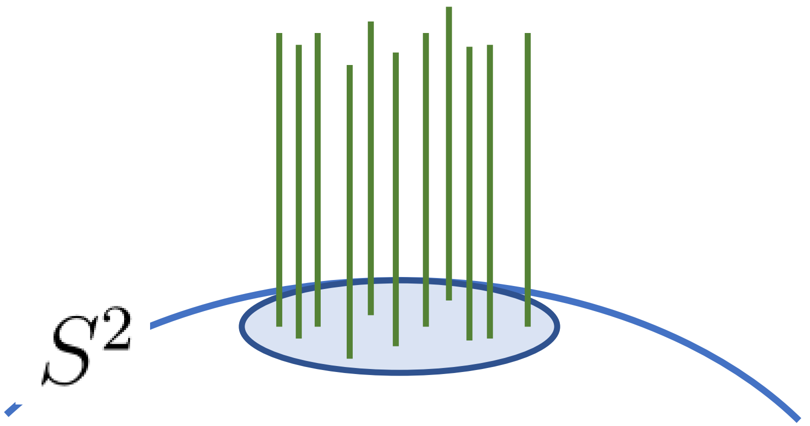
A twisted S^1 bundle on a S^2 base space

Hopf fibration

where l denotes the size of S^3.

In the region where $\theta \ll 1$,

$$ds^2 = -dt^2 + l^2(d\theta^2 + \theta^2 d\phi^2) + l^2\left(d\psi - (1 - \frac{1}{2}\theta^2)d\phi\right)^2$$



$\theta \ll 1$,

$$ds^2 = -dt^2 + l^2(d\theta^2 + \theta^2 d\phi^2) + l^2 \left(d\psi - (1 - \frac{1}{2}\theta^2) d\phi \right)^2$$

Coordinate transformation

$$x = l\theta \cos\phi, \quad y = l\theta \sin\phi, \quad z = l(\psi - \phi)$$

$$ds^2 = -dt^2 + dx^2 + dy^2 + \left(dz + \frac{1}{2l}(xdy - ydx) \right)^2$$

$$dz + \frac{1}{2l}(xdy - ydx) =: \eta$$

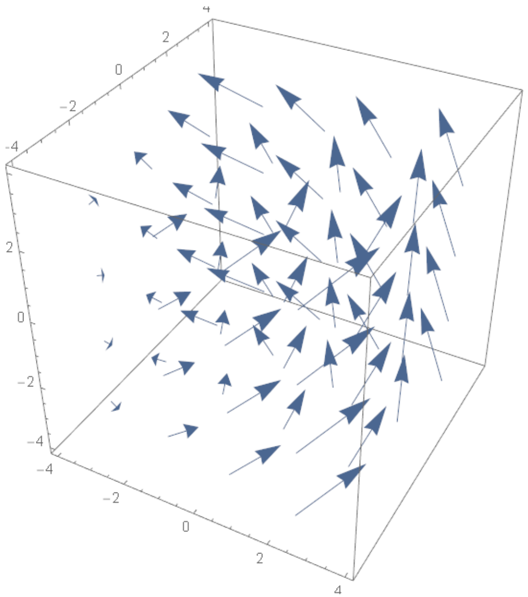
Twist of the fiber

$$\eta := dz + \frac{1}{2l}(xdy - ydx)$$

$$d\eta = \frac{1}{l} dx \wedge dy$$

$$\eta \wedge d\eta = \frac{1}{l} dx \wedge dy \wedge dz \neq 0$$

$$(\boldsymbol{\eta} \cdot \mathrm{rot}\boldsymbol{\eta} \neq 0)$$



Then, l^{-1} denotes the magnitude of twist.

Twisted magnetic field

Take a 4-dimensional gauge potential 1-form as

$$A = Bl\eta = Bl\left(dz + \frac{1}{2l}(xdy - ydx)\right)$$

where B is a constant.

Field strength

$$F = dA = Bdx \wedge dy, \qquad F_{xy} = B^z = B \quad \text{otherwize } 0$$

$$^*F = B^*(dx \wedge dy) = Bdt \wedge \eta$$

$$\boldsymbol{B} := {}^*F(\partial_t) = B\eta$$

$$d\boldsymbol{B} = B\ d\eta = \frac{B}{l}\ dx \wedge dy$$

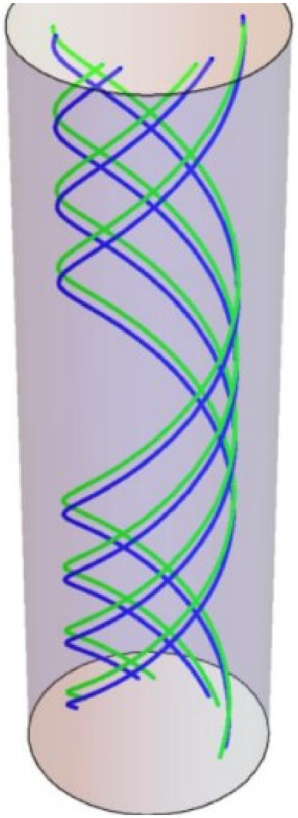
$$\boldsymbol{B} \wedge d\boldsymbol{B} = \frac{B^2}{l}\ dz \wedge dx \wedge dy \neq 0$$

Magnetic field $\boldsymbol{B}$ is twisted

Maxwell equations

$$- J_\mu dx^\mu = {}^*d\,{}^*F = \frac{B}{l}{}^*(dt \wedge dx \wedge dy) = \frac{B}{l}\eta$$

$$\mathrm{rot}\boldsymbol{B} = \boldsymbol{J} = \frac{1}{l}\boldsymbol{B}$$



$\boldsymbol{B}$ is a Beltrami magnetic field.

Background solution

$$F^{(0)}_{xy} = B, \qquad U_{(0)} = \partial_t, \qquad J_{(0)} = \frac{B}{l}\partial_z$$

$$ds^2 = -dt^2 + dx^2 + dy^2 + \left(dz + \frac{1}{2l}(xdy - ydx)\right)^2$$

Euler equation

$$\rho U^\mu_{(0)} \nabla_\mu U^\nu_{(0)} = g^{\nu\sigma} F^{(0)}_{\sigma\lambda} J^\lambda_{(0)} = g^{\nu\sigma} F^{(0)}_{\sigma z} J^z_{(0)} = 0$$

Ideal MHD

$$F^{(0)}_{\mu\nu} U^\nu_{(0)} = F^{(0)}_{\mu t} U^t_{(0)} = 0$$

Maxwell equations

$$^*d\ ^*F = \quad J_\mu dx^\mu$$

$\boldsymbol{B}$ is a homogeneous Beltrami magnetic field

Perturbations

Assumptions $$u^t = 0, \quad u^z = 0,$$

$$j^t = 0, \quad j^z = 0,$$

$$f_{xy} = 0, \quad f_{tz} = 0,$$

Nonvanishing components

$$u^x, \quad u^y, \quad j^x, \quad j^y,$$

$$f_{tx}, \quad f_{ty}, \quad f_{zx}, \quad f_{zy}$$

These variables are assumed to be dependent on t and z.

$$\boldsymbol{e} := \begin{pmatrix} e_x \\ e_y \end{pmatrix} = \begin{pmatrix} f_{xt} \\ f_{yt} \end{pmatrix} \qquad \boldsymbol{b} := \begin{pmatrix} b_x \\ b_y \end{pmatrix} = \begin{pmatrix} f_{yz} \\ f_{zx} \end{pmatrix}$$

Euler equation

$$\rho U^t \nabla_t (U^\nu_{(0)} + u^\nu) = g^{\nu\sigma}(F^{(0)}_{\sigma\lambda} + f_{\sigma\lambda})(J^\lambda_{(0)} + j^\lambda)$$

$$\rho \nabla_t u^\nu = g^{\nu\sigma}(F^{(0)}_{\sigma\lambda} j^\lambda + f_{\sigma\lambda} J^\lambda_{(0)})$$

$$\nu = x, y$$

$$\rho \partial_t u^I = B(\epsilon^I{}_K j^K + \frac{1}{l} f^I{}_z)$$

Euler eqution

$$j^I = -\frac{\rho}{B}\epsilon^I{}_J \partial_t u^J + \frac{1}{l}\epsilon^{IJ} f_{Jz}$$

$$I, J = x, y$$

Euler equations: $j^I = -\frac{\rho}{B}\epsilon^I{}_J \partial_t u^J + \frac{1}{l}\epsilon^{IJ} f_{Jz}$

MHD conditions: $u^I = -\frac{1}{B}\epsilon^{IJ} f_{tJ}$

Faraday's law: $\partial_t f_{zI} = \partial_z f_{tI}$

Ampere-Maxwell's law: $j^I = -\partial_t f^{tI} - \partial_z f^{zI}$ $\quad (I, J = x, y)$

Erasing j and u, we have wave equations.

$$-\frac{1}{c_a^2}\partial_t^2 f_{AI} + \partial_z^2 f_{AI} = \frac{1}{l}\epsilon_I{}^J \partial_z f_{AJ}$$

$$\partial_t f_{zI} = \partial_z f_{tI}$$

$$-\frac{1}{c_a^2}\partial_t^2 u^J + \partial_z^2 u^J = \frac{1}{l}\epsilon^J{}_K \partial_z u^K$$

$$\frac{1}{c_a^2} = 1 + \frac{\rho}{B^2} \quad : \text{Alfvén velocity}$$

$$\left(-\frac{1}{c_a^2}\partial_t^2 + \partial_z^2\right)\boldsymbol{f}_A - \frac{1}{l}\partial_z(\boldsymbol{\epsilon}\cdot\boldsymbol{f}_A) = 0$$

$$\boldsymbol{f}_t = \boldsymbol{e} = \begin{pmatrix} e_x \\ e_y \end{pmatrix} \qquad \boldsymbol{f}_z = \boldsymbol{b} = \begin{pmatrix} b_x \\ b_y \end{pmatrix}$$

Wave solutions

$$\boldsymbol{e}^{\pm} = \boldsymbol{f}_t^{\pm} = \begin{pmatrix} 1 \\ \pm i \end{pmatrix} \mathcal{E} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \qquad \boldsymbol{b}^{\pm} = \boldsymbol{f}_z^{\pm} = \begin{pmatrix} \mp i \\ 1 \end{pmatrix} \mathcal{B} e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$$

$$\boldsymbol{k}\cdot\boldsymbol{x} := kz - \omega t$$

$$\left(-\frac{1}{c_a^2}\partial_t^2 + \partial_z^2\right)\boldsymbol{f}_A - \frac{1}{l}\partial_z(\boldsymbol{\epsilon}\cdot\boldsymbol{f}_A) = 0$$

Dispersion relation

$$\frac{1}{c_a^2}\omega_{\pm}^2 = k^2 \mp \frac{1}{l}k \qquad v_p^{\pm} = \frac{\omega_{\pm}}{k} = c_a\sqrt{1 \mp \frac{1}{lk}}$$

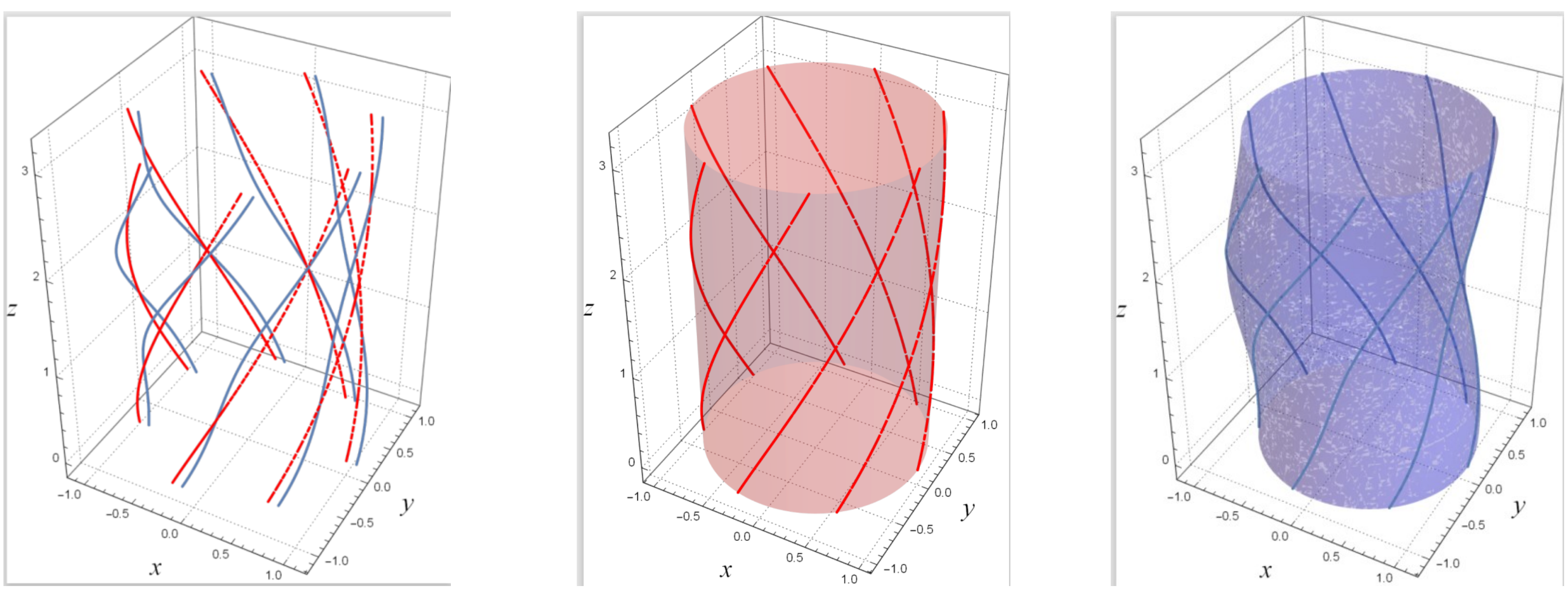
角振動数が実になるために
片方の偏光の波数に下限が現れる

$$v_g^{\pm} = \frac{d\omega_{\pm}}{dk} = c_a$$

$$\boldsymbol{e}^{\pm} = \Re\left[\begin{pmatrix}1\\ \pm i\end{pmatrix}\mathcal{E}e^{i(kz-\omega_{\pm}t)}\right] = \mathcal{E}\begin{pmatrix}\cos(kz-\omega_{\pm}t)\\ \mp\sin(kz-\omega_{\pm}t)\end{pmatrix}$$

$$\boldsymbol{b}^{\pm} = \Re\left[\begin{pmatrix}\mp i\\ 1\end{pmatrix}\mathcal{B}e^{i(kz-\omega_{\pm}t)}\right] = \mathcal{B}\begin{pmatrix}\pm\sin(kz-\omega_{\pm}t)\\ \cos(kz-\omega_{\pm}t)\end{pmatrix}$$

$(\boldsymbol{e}^{\pm}, \boldsymbol{b}^{\pm})$ describes a right-handed (left-handed) circularly polarized wave.



Perturbation of magnetic field

For a fixed time

$$\boldsymbol{b}^{\pm} = \mathcal{B}(\pm \sin kz \ dx + \cos kz \ dy)$$

$$d\boldsymbol{b}^{\pm} = \mathcal{B}k(\mp \cos kz \ dz \wedge dx - \sin kz \ dz \wedge dy)$$

$$\boldsymbol{b}^{\pm} \wedge d\boldsymbol{b}^{\pm} = \mp \mathcal{B}^2 k \ dx \wedge dz \wedge dy$$

$\boldsymbol{b}$ is twisted.

$$\mathrm{rot}\ \boldsymbol{b}^{\pm} = \mp k \boldsymbol{b}^{\pm}$$

$$\mathrm{rot}\ \boldsymbol{e}^{\pm} = \mp k \boldsymbol{e}^{\pm}$$

$\boldsymbol{b}$ and $\boldsymbol{e}$ are Beltrami fields on time slices.

$\boldsymbol{b}(t, z)$ and $\boldsymbol{e}(t, z)$ are rotating Beltrami fields.

Rotation of polarization

Superposition of right-handed and left handed modes

$$\frac{1}{2}(\boldsymbol{e}^+ + \boldsymbol{e}^-) = \frac{1}{2}\mathcal{E}\begin{pmatrix}1\\ i\end{pmatrix}e^{i(kz-\omega_+ t)} + \frac{1}{2}\mathcal{E}\begin{pmatrix}1\\ -i\end{pmatrix}e^{i(kz-\omega_- t)}$$

$$= \mathcal{E}\begin{pmatrix}\cos\Delta\omega t\\ \sin\Delta\omega t\end{pmatrix}e^{i(kz-\bar{\omega}t)}$$

$$\bar{\omega} := \frac{1}{2}(\omega_+ + \omega_-).$$

$$\Delta\omega := \frac{1}{2}(\omega_+ - \omega_-).$$

$$= \mathcal{E}\begin{pmatrix}\cos\frac{c_a}{2l}t\\ \sin\frac{c_a}{2l}t\end{pmatrix}e^{ik(z-c_a t)}$$

$lk \gg 1$ case

$$\omega_\pm = c_a k\sqrt{1 \pm \frac{1}{lk}} \simeq c_a k(1 \pm \frac{1}{2lk})$$

Plane of polarization rotates

Energy and momentum

Energy momentum tensor of the Alfvén waves

$$T_{\mu\nu} = g^{\alpha\beta} f_{\mu\alpha} f_{\nu\beta} - \frac{1}{4} g_{\mu\nu} \left(g^{\alpha\beta} g^{\lambda\sigma} f_{\alpha\lambda} f_{\beta\sigma} \right)$$

4-momentum

$$P_\mu := T_{\mu\nu} \xi_t^\nu = T_{\mu t}$$

Killing vector

$$\xi_t^\nu = \partial_t = (1, 0, 0, 0)$$

$$P_t = f_{tx}^2 + f_{ty}^2 + \frac{1}{2} \left(-f_{tx}^2 - f_{ty}^2 + f_{zx}^2 + f_{zy}^2 \right)$$

$$= \frac{1}{2} \left(\mathcal{E}^2 + \mathcal{B}^2 \right)$$

$$P_z = f_{zx} f_{tx} + f_{zy} f_{ty}$$

$$= e_x b_y - b_x e_y = [\boldsymbol{e} \times \boldsymbol{b}]_z = \mathcal{E}\mathcal{B}$$

Angular momentum

Killing vector $\xi^\nu_\varphi = x\partial_y - y\partial_x = (0, -y, x, 0)$

Angular momentum

$$L_\mu := T_{\mu\nu}\xi^\nu_\varphi = -yT_{\mu x} + xT_{\mu y}$$

$$L_{(\phi)} = -yT_{tx} + xT_{ty}$$
$$= \mathcal{E}\mathcal{B}\left(\frac{y^2}{2l}\sin^2(kz-\omega t) + \frac{x^2}{2l}\cos^2(kz-\omega t)\right)$$

Take average on t and z

$$< L_{(\phi)} > = \mathcal{E}\mathcal{B}\left(\frac{y^2}{2l} < \sin^2(kz-\omega t) > + \frac{x^2}{2l} < \cos^2(kz-\omega t) >\right)$$
$$= \frac{1}{2}\mathcal{E}\mathcal{B}\left(\frac{y^2}{2l} + \frac{x^2}{2l}\right) = \frac{r^2}{4l}\mathcal{E}\mathcal{B}$$

Alfvén waves carry angular momentum!

Summary

- Using a small domain of S^3, twisted S^1 bundle on S^2 base space, we constructed a homogeneous Beltrami magnetic field.
- On the assumption of ideal MHD approximation, we investigated Alfven waves propagate on the Beltrami magnetic field.

Summary

- We obtain wave equation for Alfven waves in a Beltrami magnetic field that admits left-handed and right-handed circular polarized wave solutions.
- Left-right symmetry is broken by the twisted background magnetic field, then the dispersion relations for these modes are different.
- Cut-off frequency for one of the mode exists.
- Superposing these two modes, we have wave solutions such that plane of polarization rotates.
- Wave front is not perpendicular to the magnetic field.
- Alfven waves carry energy and angular momentum.