Finite N superconformal index via the AdS/CFT correspondence

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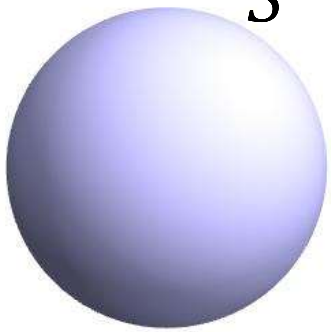
[Arai, YI, arXiv:1904.09776]
[YI, arXiv:2108.12090]

<u>AdS/CFT correspondence</u>

[Maldacena, hep-th/9711200]

$\mathcal{N}$=4 U(N) SYM ⟺ $\mathrm{AdS}_5 \times$ S^5



Gauge invariant operators

Various objects

We use the superconformal index as a mathematical tool to express the spectrum.

Superconformal index

[Kinney, Maldacena, Minwalla, Raju, hep-th/0510251]

Cartan generators of PSU(2,2|4)

H : Hamiltonian (Dilatation)
J_1, J_2 : Angular momenta
R_1, R_2, R_3 : R-charges

$$I(q, y, u_i) = \mathrm{Tr}_{S^3 \times R}\left[(-1)^F q^{H+\frac{J_1+J_2}{2}} y^{J_1-J_2} u_1^{R_1} u_2^{R_2} u_3^{R_3}\right]$$

$$(u_1 u_2 u_3 = 1)$$

We can calculate this quantity on the gauge theory side for an arbitrary N by using localization formula.

Examples

(We turn off variables except for q (to save the space).)

$$I_{U(1)} = 1 + 0q^{\frac{1}{2}} + 3q - 2q^{\frac{3}{2}} + 3q^2 + 0q^{\frac{5}{2}} + 0q^3 + 6q^{\frac{7}{2}} - 6q^4$$
$$+ 0q^{\frac{9}{2}} + 12q^5 - 18q^{\frac{11}{2}} + 27q^6 - 12q^{\frac{13}{2}} - 27q^7$$
$$+ 60q^{\frac{15}{2}} - 60q^8 + 24q^{\frac{17}{2}} + 76q^9 - 174q^{\frac{19}{2}} + 162q^{10}$$
$$+ 0q^{\frac{21}{2}} - 240q^{11} + 432q^{\frac{23}{2}} - 348q^{12} - 144q^{\frac{25}{2}}$$
$$+ 783q^{13} \ldots$$

$$I_{U(\infty)} = 1 + 0q^{\frac{1}{2}} + 3q - 2q^{\frac{3}{2}} + 9q^2 - 6q^{\frac{5}{2}} + 21q^3 - 18q^{\frac{7}{2}} + 48q^4$$
$$-42q^{\frac{9}{2}} + 99q^5 - 96q^{\frac{11}{2}} + 200q^6 - 198\, q^{\frac{13}{2}} + 381q^7$$
$$-396q^{\frac{15}{2}} + 711q^8 - 750q^{\frac{17}{2}} + 1278q^9 - 1386q^{\frac{19}{2}} + 2256q^{10}$$
$$-2472q^{\frac{21}{2}} + 3879q^{11} - 4320q^{\frac{23}{2}} + 6564q^{12} - 7362\, q^{\frac{25}{2}}$$
$$+10890q^{13} \quad \ldots$$

Large N

The parameter relation

$$N = \frac{L_{\text{AdS}}^4}{l_p^4}$$

L_{AdS}: AdS radius

l_p: Planck length

Large N $\leftrightarrow$ Classical analysis is justified.

$I_{U(\infty)}$ =

Supergravity KK modes in S^5

[Kinney, Maldacena, Minwalla, Raju, hep-th/0510251]

Finite N

Parameter relations

L_{AdS}: AdS radius

l_p: Planck length

$$N = \frac{L_{\mathrm{AdS}}^4}{l_p^4}$$

finite N → quantum gravity

T_{D3}: D3 tension

$$N = L_{\mathrm{AdS}}^4 T_{\mathrm{D3}}$$

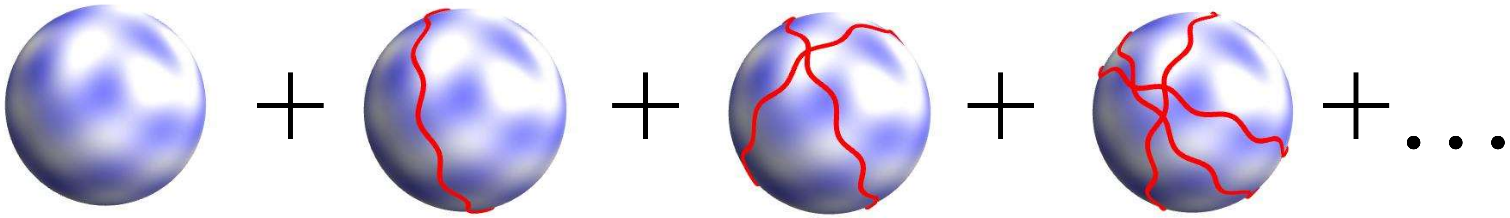
finite N → Expanded D3-branes (Giant gravitons)

Interesting possibility

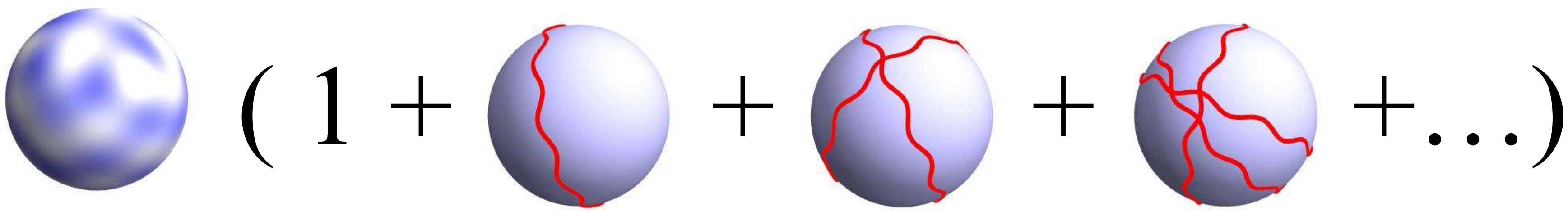
If a quantity is protected from quantum gravity correction, it may be possible to reproduce the finite N correction to the quantity as D3-brane contributions.

Superconformal index seems to be such a quantity.

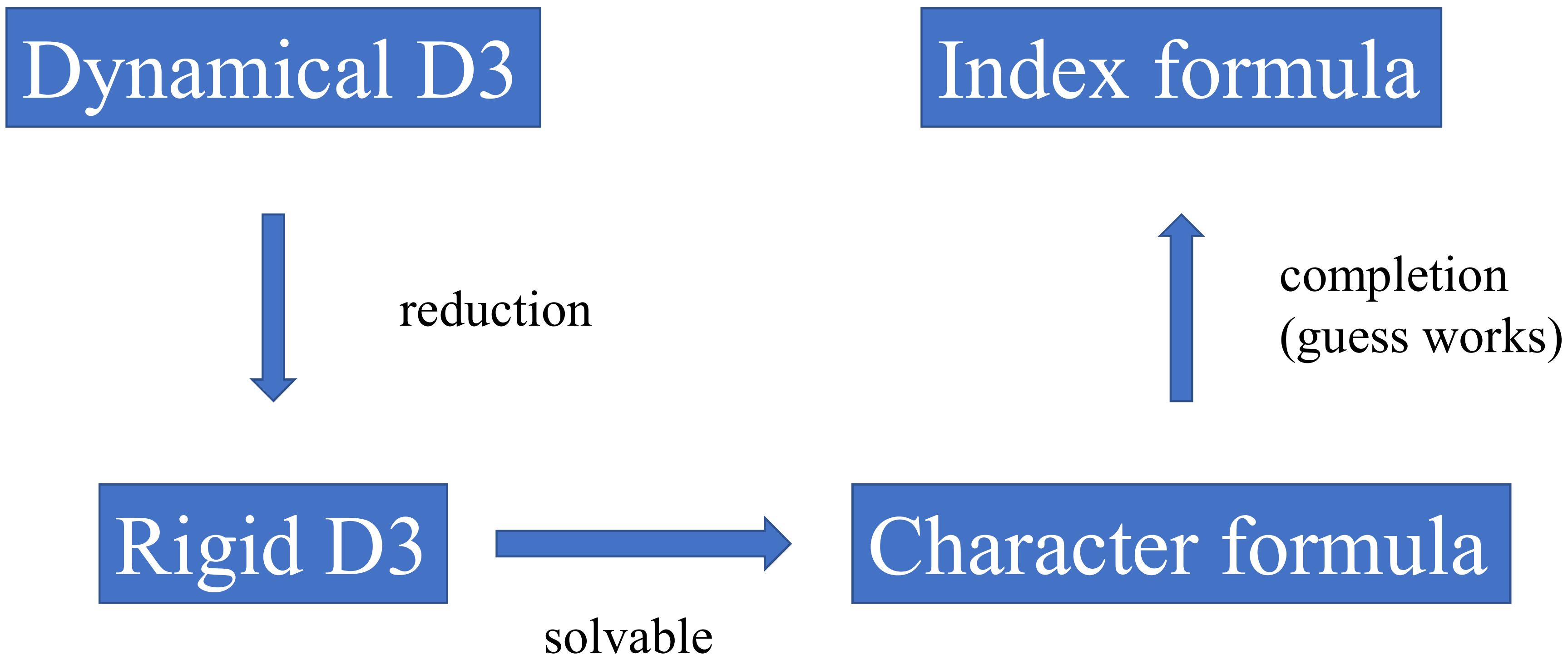
Main claim [YI, arXiv:2108.12090]

$I_{U(N)} =$ 

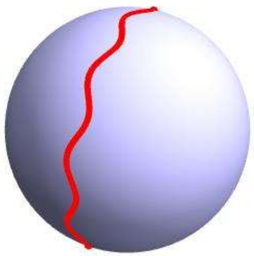
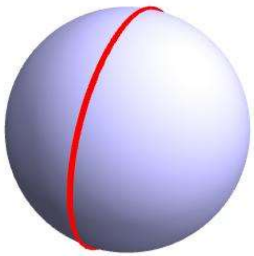
or, equivalently,

$=$ 

Strategy


Dynamical D3
reduction
Rigid D3
solvable
Character formula
completion
(guess works)
Index formula

Rigid branes (a toy model)

 =  + ~~excitations~~

Rigid brane = D3 wrapped on a large S^3 in S^5

$$az_1 + bz_2 + cz_3 = 0 \qquad (a, b, c) \in \mathrm{CP}^2$$

A rigid D3 = A point particle in CP^2

Degenerate states in $[N, 0]$ of $SU(3) \in SO(6)_R$

“Index” of a rigid D3

$$I = q^N \chi_{[N,0]}(u)$$

q^N: the energy of D3

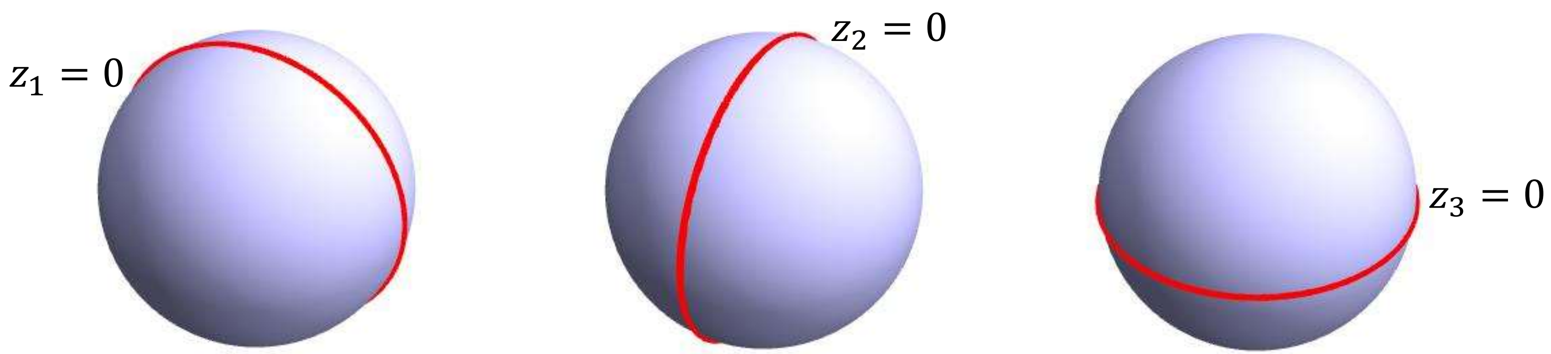
$\chi_{[N,0]}(u)$: SU(3) character

$$\chi_{[N,0]}(u) = \sum_{k_1+k_2+k_3=N} u_1^{k_1} u_2^{k_2} u_3^{k_3}$$

$$\chi_{[0,0]} = 1$$
$$\chi_{[1,0]} = u_1 + u_2 + u_3$$
$$\chi_{[2,0]} = u_1^2 + u_2^2 + u_3^2 + u_1 u_2 + u_2 u_3 + u_3 u_1$$
$$\vdots$$

Decomposition to harmonic oscillators

$$q^N \chi_{[N,0]} = \frac{(qu_1)^N}{\left(1-\frac{u_2}{u_1}\right)\left(1-\frac{u_3}{u_1}\right)} + \frac{(qu_2)^N}{\left(1-\frac{u_3}{u_2}\right)\left(1-\frac{u_1}{u_2}\right)} + \frac{(qu_3)^N}{\left(1-\frac{u_1}{u_3}\right)\left(1-\frac{u_2}{u_3}\right)}$$



$(qu_i)^N$: classical energy and charges of a brane wrapped around a large circle

$\frac{1}{\left(1-\frac{u_{i+1}}{u_i}\right)\left(1-\frac{u_{i-1}}{u_i}\right)}$: harmonic oscillators of two rigid motions

Completion

Rigid motions
(only two d.o.f.)

$$(qu_3)^N \frac{1}{\left(1-\frac{u_1}{u_3}\right)\left(1-\frac{u_2}{u_3}\right)}$$

→

All fluctuations

$$(qu_3)^N \frac{\left(1-\frac{q^{\frac{1}{2}}y}{u_3}\right)\left(1-\frac{q^{\frac{1}{2}}}{yu_3}\right)\cdots}{\left(1-\frac{1}{qu_3}\right)\left(1-\frac{u_1}{u_3}\right)\left(1-\frac{u_2}{u_3}\right)\cdots} =: F_3$$

↑obtained from the mode expansion of the vector multiplet on the wrapped D3

This is the index of the U(1) gauge theory on a wrapped D3.

We also define F_1 and F_2 in the same way.

$$I_{\text{single}} = I_{\text{SUGRA}} \times (F_1 + F_2 + F_3)$$

Numerical check

We want to get

$$I_{U(1)} = 1 + 0q^{\frac{1}{2}} + 3q - 2q^{\frac{3}{2}} + 3q^2 + 0q^{\frac{5}{2}} + 0q^3 + 6q^{\frac{7}{2}} - 6q^4 + 0q^{\frac{9}{2}} + 12q^5 - 18q^{\frac{11}{2}} + 27q^6 - 12q^{\frac{13}{2}} - 27q^7 + \cdots$$

SUGRA gives

$$I_{\text{SUGRA}} = 1 + 0q^{\frac{1}{2}} + 3q - 2q^{\frac{3}{2}} + 9q^2 - 6q^{\frac{5}{2}} + 21q^3 - 18q^{\frac{7}{2}} + 48q^4 - 42q^{\frac{9}{2}} + 99q^5 - 96q^{\frac{11}{2}} + 200q^6 - 198\,q^{\frac{13}{2}} + 381q^7 + \cdots$$

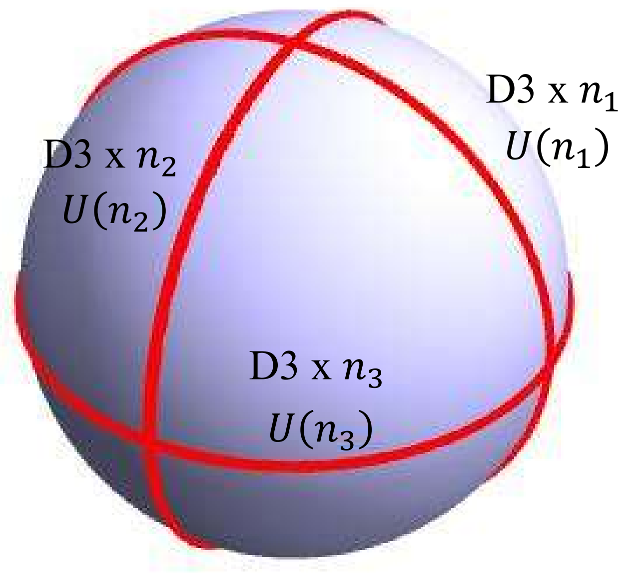
Inclusion of I_{single} [Arai, YI, arXiv:1904.09776]

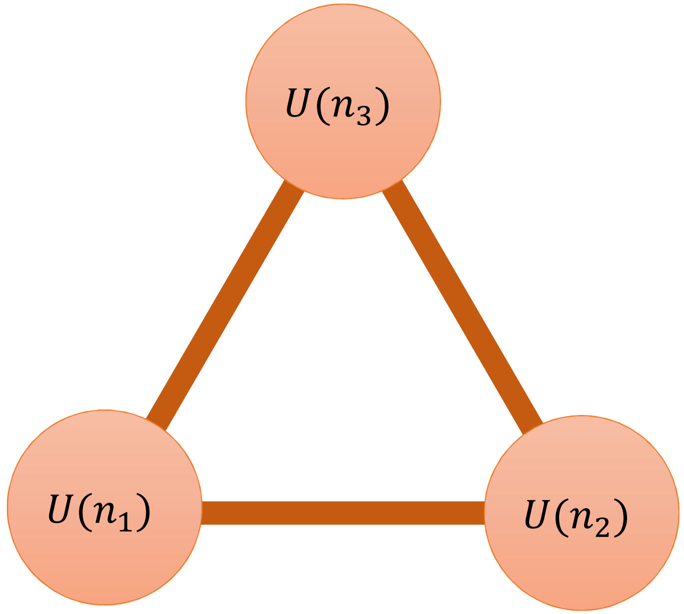
$$I_{\text{SUGRA}} + I_{\text{single}} = 1 + 0q^{\frac{1}{2}} + 3q - 2q^{\frac{3}{2}} + 3q^2 + 0q^{\frac{5}{2}} + 0q^3 + 6q^{\frac{7}{2}} - 6q^4 + 0q^{\frac{9}{2}} + 12q^5 - 18q^{\frac{11}{2}} - 85q^6 + 504\,q^{\frac{13}{2}} - 1896q^7 + \cdots$$

This is encouraging, but still we have mismatch.

2^{nd} completion

Let us include the contribution from multiple-wrapping contributions.





$U(n_1) \times U(n_2) \times U(n_3)$ quiver gauge theory is realized on the brane system

Proposal

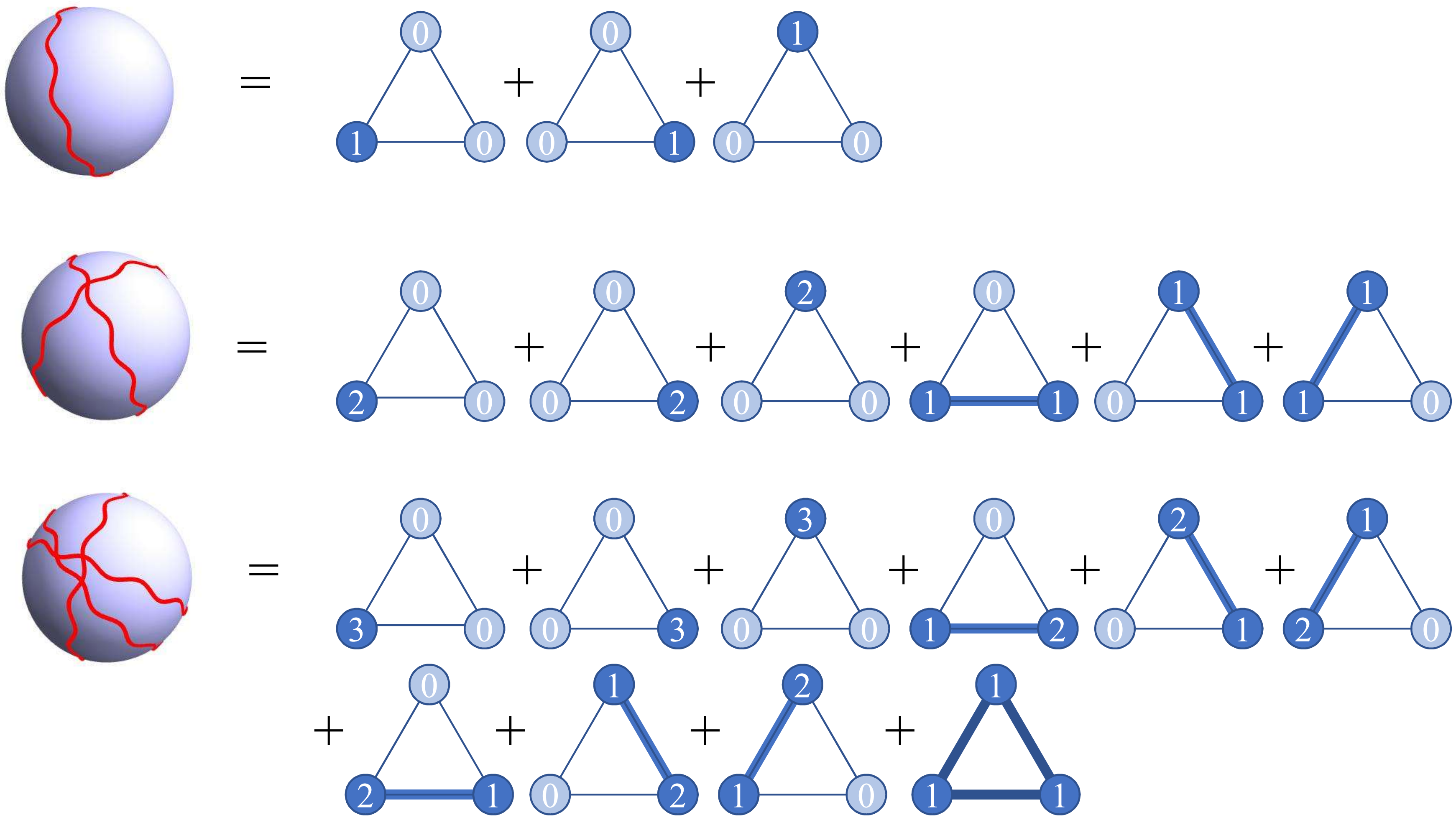
The index is given by the following formula. [YI, arXiv:2108.12090]

$$I_{U(N)} = I_{\mathrm{SUGRA}} \sum_{n_1,n_2,n_3=0}^{\infty} q^{(n_1+n_2+n_3)N} u_1^{n_1 N} u_2^{n_2 N} u_3^{n_3 N} F_{n_1,n_2,n_3}$$

F_{n_1,n_2,n_3}: index of the quiver gauge theory on the D3-branes (N-indep)

I_{single} : (1,0,0), (0,1,0), (0,0,1)
I_{double} : (2,0,0), (0,2,0), (0,0,2), (1,1,0), (1,0,1), (0,1,1)
I_{triple} : (3,0,0),(0,3,0),(0,0,3),(2,1,0),(1,2,0),(0,2,1),(0,1,2),(2,0,1),(1,0,2),(1,1,1)
…

Numerical check



Numerical check

Inclusion of multiple-wrapping contributions gives (for $N = 1$)

$$I_{\text{SUGRA}} + I_{\text{single}} + I_{\text{double}} + I_{\text{triple}} + \cdots$$
$$= 1 + 0q^{\frac{1}{2}} + 3q - 2q^{\frac{3}{2}} + 3q^2 + 0q^{\frac{5}{2}} + 0q^3 + 6q^{\frac{7}{2}} - 6q^4$$
$$+ 0q^{\frac{9}{2}} + 12q^5 - 18q^{\frac{11}{2}} + 27q^6 - 12q^{\frac{13}{2}} - 27q^7$$
$$+ 60q^{\frac{15}{2}} - 60q^8 + 24q^{\frac{17}{2}} + 76q^9 - 174q^{\frac{19}{2}} + 162q^{10}$$
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$$+ 783q^{13} \dots = I_{U(1)}$$

By summing up contributions up to $n_1 + n_2 + n_3 \leq 3$ we found complete agreement up to q^{13} for $N = 1$. [YI, arXiv:2108.12090]

Summary

- We proposed a new method to calculate the index of N=4 SYM on the AdS side.
- Although we have no proof at present, numerical analysis showed it reproduces the correct answer.
- The method has been applied to many examples of AdS/CFT correspondence, and works well.

Application to other systems

AdS	CFT	Large N	Single-wrapping	Multiple-wrapping
$AdS_5 \times S^5$	4d $\mathcal{N}$=4 SYM	[Kinney, Maldacena, Minwalla, Raju, 05]	✔[Arai, YI, 19]	✔[Arai, Fujiwara, YI, Mori, 20][YI, 21]
$AdS_5 \times S^5/Z_k^S$	4d $\mathcal{N}$=3	[IY, Yokoyama, 16]	✔[Arai, YI, 19]	
$AdS_5 \times S^5/\Gamma$	4d $\mathcal{N}$=1 quiver	[Nakayama, 05]	✔[Arai, Fujiwara, YI, Mori, 19]	
$AdS_5 \times SE_5$	4d $\mathcal{N}$=1 quiver	[Nakayama, 06][Eager, Schmude, Tachikawa, 12][Agarwal, Amariti, Mariotti 13]	✔[Arai, Fujiwara, YI, Mori, 19]	(✔) [Fujiwara, in preparation]
$AdS_5 \times S^5_\alpha$	4d $\mathcal{N}$=2 AD & MN	[Fayyazuddin, Spalinski, 98][Aharony, Fayyazuddin, Maldacena, 98]	✔[YI, Murayama, 21]	
$AdS_4 \times S^7/Z_k$	3d ABJM	[Bhattacharya, Bhattacharyya, Minwalla, Raju, 08][Kim, 09]	✔[Arai, Fujiwara, YI, Mori, Yokoyama, 20]	
$AdS_7 \times S^4$	6d (2,0)	[Bhattacharya, Bhattacharyya, Minwalla, Raju, 08]	✔[Arai, Fujiwara, YI, Mori, Yokoyama, 20]	(✔) [Arai, Fujiwara, YI, Mori, Yokoyama]
$AdS_7 \times S^4/Z_k$	6d (1,0)	[Ahn, Oh, Tatar, 98]	✔[Fujiwara, YI, Mori, 21]	

As far as we have checked the formula reproduces finite N index correctly!

Thank you