Penrose Limit:
A Stringy Regime in Holography

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B. Du, MH, JHEP 03 (2021), 246, arXiv:2101.07484;
B. Du, MH, JHEP 08 (2021), 006, arXiv:2104.12502.

- The AdS/CFT correspondence (Maldacena, 1997) is a deep idea which relates two seemingly totally different theories, namely type IIB string theory or supergravity on AdS background and the $\mathcal{N} = 4$ $SU(N)$ super-Yang-Mills theory.

- Although the correspondence has found flourishing applications in many topics, the precise quantitative tests of the holographic dictionary are mostly restricted to supersymmetry protected quantities in the supergravity approximation, such as the spectrum and correlation functions of BPS operators.

- Without an alternative effective method to handle string theory in the deeply stringy regime, a common perspective is to simply take the super-Yang-Mills theory as a non-perturbative definition of AdS string theory at any finite coupling and energy scale, assumed to be valid unless otherwise convincingly explicitly contradicted.

- A particularly interesting avenue for progress in the precise tests of the holographic correspondence in the stringy regime is to take a Penrose limit of the type IIB $AdS_5 \times S^5$ background.

- The geometry becomes a pp-wave or plane wave background Blau et al 2001 with also maximal supersymmetry

$$ds^2 = -4dx^+dx^- - \mu^2(\vec{r}^{\,2} + \vec{y}^{\,2})(dx^+)^2 + d\vec{r}^{\,2} + d\vec{y}^{\,2}, \quad (1)$$

where x^+, x^- are light cone coordinates, $\vec{r}, \vec{y}$ are 4-vectors, and the parameter μ measures the spacetime curvature as well as the Ramond-Ramond flux $F_{+1234} = F_{+5678} \sim \mu$.

- The free string spectrum can be obtained in the light cone gauge using Green-Schwarz formalism similar to the flat space. (Unlike AdS space)

- In the groundbreaking paper, Berenstein, Maldacena and Nastase (BMN 2002) proposed a type of near-BPS operators, which correspond to the type IIB closed strings on the pp-wave background. The free string spectrum is correctly reproduced by gauge interactions as the planar conformal dimensions of BMN operators.

- The BMN scaling limit with large R-charge $J \sim \sqrt{N} \sim \infty$ appears to be the right Goldilocks limit in this situation.
  A smaller R-charge would not provide finite string interactions in the strict $N \sim \infty$ limit,
  while a larger R-charge may blow up the strings into D-branes, known as giant gravitons, studied in early papers e.g. McGreevy:2000, Hashimoto:20 Balasubramanian:2001, Corley:2001.

- This appears to be a promising ground for quantitative explorations of the holographic duality in stringy regimes, as the dual theories on both sides can be either free or weakly coupled.
  Caveat: spacetime highly (even infinitely) curved!

More Physical Motivation

- A main goal of the theories of quantum gravity is to understand the physics in the regime beyond the reach of classical gravity, e.g. in the highly curved spacetime region near the black hole singularity.

- For AdS_d space with $d > 2$, the scalar curvature is negative $R = -\frac{d(d-1)}{r^2}$ where r is the radius of the AdS space, so the spacetime is highly curved if the radius r is very small (compared to the string or Planck length).

- For the pp-wave background, the scalar curvature actually vanishes, while the non-vanishing component of the Ricci curvature is proportional to μ^2. We will focus on the $\mu \sim \infty$ or infinite curvature limit.

- Certainly the effective action of classical gravity completely breaks down in this case, but this is not necessarily a bad situation as we can instead probe the fundamental nature of spacetime in a stringy regime.
The classical geometry blows up and is very singular, but the quantum theory is completely finite and consistent.
To quote a classic movie, "I've a feeling we're not in Kansas anymore."

- The Penrose limit provides a new twist to the holography story. In the celebrated AdS/CFT holographic dictionary Witten:1998, the CFT lives at the boundary of a bulk AdS space and its local operators couple to the boundary configurations of the AdS bulk fields.

- However, although the pp-wave background comes from a Penrose limit of the AdS space, the geometry is rather different. As such, it is not clear how to directly apply the standard AdS holographic dictionary, particularly in the situations with finite string interactions.

- Our approach in some previous papers Huang:2002a, Huang:2002b, Huang:2010, Huang:2019a, Huang:2019b is to consider another corner of the parameter space in the BMN limit, focusing on the free gauge theory. In this case, the string theory side becomes infinitely curved $\mu \sim \infty$, and strings are effectively infinitely long and tensionless, but can still have finite string interactions.

- Most interestingly, since the string spectrum is completely degenerate, the tensionless string can jump from one excited state to another without energy cost through a quantum unitary transition. It turns out that in this case the effective string coupling constant should be identified with a finite genus counting parameter $g := \frac{J^2}{N}$.

- Since the full fledged holographic dictionary is no longer available in the pp-wave background, our pragmatic approach is to try to compute the physical quantities on both sides of the correspondence and find potential non-trivial agreements.

- In this sense, a mismatch with naive expectation is not necessarily a contradiction of the holographic principle. Instead, one should focus on finding aspects where the calculations from both side do match, and try to give physical derivations or proofs of such mathematical coincidence.

- Besides the free string spectrum originally considered in BMN 2002 (0th entry), there are some more tests of the pp-wave holography. Three entries (we will refer to them as the 1st, 2nd, 3rd entries) of "pp-wave holographic dictionary":

  1. The free planar three-point functions of BMN operator should correspond to the Green-Schwarz light cone string field cubic vertex in the infinitely curved pp-wave background. Huang:2002a

  2. In the papers Huang:2002b, Huang:2010, we further proposed a factorization formula, where the free (higher genus) BMN correlators are holographically related to string (loop) diagram calculations by pasting together the cubic string vertices without propagator.

  3. More recently, we propose a probability interpretation of the BMN two-point functions Huang:2019, with indirect tests through mostly the non-negativity of BMN two-point functions. This entry (focus of this talk) seems most interesting, though the evidence is also relatively the weakest.

Some Notations

- The BMN vacuum operator is simply proportional Tr (Z^J) where Z is a complex scalar in the $\mathcal{N} = 4$ $SU(N)$ super-Yang-Mills theory. One can insert the four remaining real scalars into the trace with phases, corresponding to string modes in four of the eight transverse dimensions.

- The BMN operators are then denoted as $O^J_{(m_1, m_2, \cdots, m_k)}$, where the positive and negative integer modes represent the left and right moving stringy excited modes, while the zero modes are supergravity modes representing discretized momenta in the corresponding traverse direction.

- Due to the closed string level matching condition

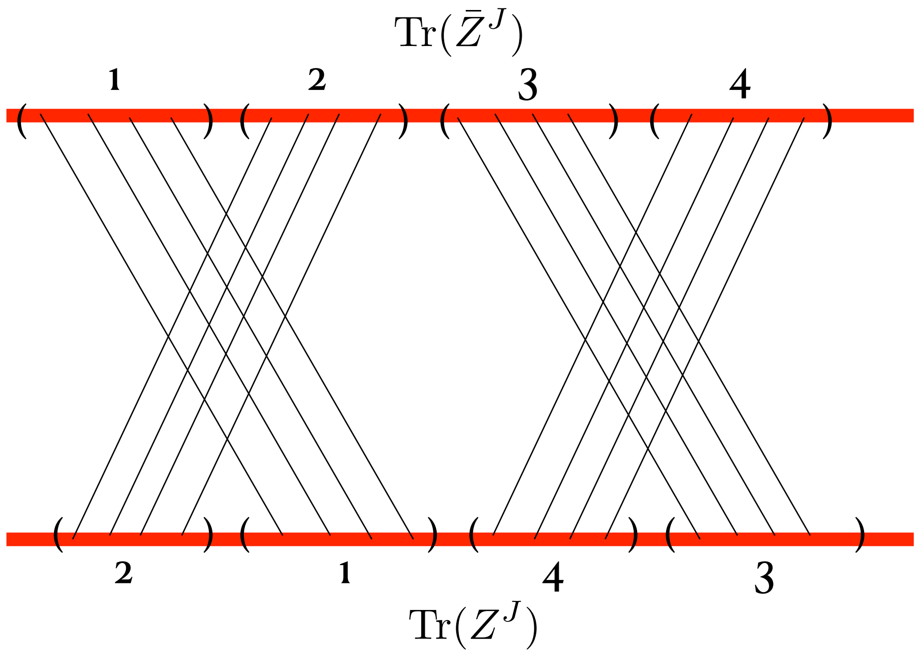
$$\sum_i m_i = 0, \tag{2}$$

the excited stringy states have at least two string modes.

- The BMN operators are properly normalized to be orthonormal at planar level, and the genus h two-point functions are proportional to g^{2h} as

$$\begin{aligned} &\langle \bar{O}^J_{(m_1,m_2,\cdots,m_k)} O^J_{(n_1,n_2,\cdots,n_k)} \rangle_0 = \delta_{m_1,n_1} \cdots \delta_{m_k,n_k}, \\ &\langle \bar{O}^J_{(m_1,m_2,\cdots,m_k)} O^J_{(n_1,n_2,\cdots,n_k)} \rangle_h \sim g^{2h}. \end{aligned} \tag{3}$$

- An example of torus (genus one) diagram



The Probability Interpretation

- The BMN two-point functions are real and symmetric, and there is a nice normalization relation summing over one set of mode numbers

$$\sum_{\sum_{i=1}^{k} n_k = 0} \langle \bar{O}^J_{(m_1,m_2,\cdots,m_k)} O^J_{(n_1,n_2,\cdots,n_k)} \rangle_h = \frac{g^{2h}}{2^{2h}(2h+1)!}. \tag{4}$$

- We define a matrix element, summing up all genus contributions with a proper normalization by the all-genera formula of vacuum correlator

$$p_{(m_1,m_2,\cdots,m_k),(n_1,n_2,\cdots,n_k)} = \frac{g}{2\sinh(\frac{g}{2})} \sum_{h=0}^{\infty} \langle \bar{O}^J_{(m_1,m_2,\cdots,m_k)} O^J_{(n_1,n_2,\cdots,n_k)} \rangle_h, \tag{5}$$

so that it looks like a probability distribution

$$\sum_{\sum_{i=1}^{k} n_k = 0} p_{(m_1,m_2,\cdots,m_k),(n_1,n_2,\cdots,n_k)} = 1. \tag{6}$$

- To interpret the matrix elements as a probability distribution, they need to be non-negative.

- For the case of two string modes, the non-negativity at any genus can be easily proven since the two string modes are opposite numbers Huang:2019.

- For the case of four string modes, it turns out that the genus one two-point functions can be negative. There seems to be a rule forbidding the "crowdedness" of string modes, that we can not holographically use up all four remaining scalars to fully occupy the four transverse dimensions with $SO(4)$ rotational symmetry unbroken by the Ramond-Ramond flux in the pp-wave background.

- For the case of three string modes, we are not aware of a simple analytic proof of the non-negativity. Instead, we perform the detailed calculations and explicitly verify the non-negativity up to genus two.

A Novel Entry

- The results suggest a novel entry (the 3rd entry) of the pp-wave holographic dictionary

$$p_{(m_1,\cdots,m_k),(n_1,\cdots,n_k)} = |\langle m_1,\cdots,m_k|\widehat{U}(g)|n_1,\cdots,n_k\rangle|^2, \quad k=2,3, \quad (7)$$

where the operator $\widehat{U}(g)$ describes the quantum unitary transition between the degenerate tensionless strings. The BMN single string states form a complete orthonormal basis of the Hilbert space under such finite string interactions

$$\sum_{\sum_{i=1}^{k} n_k = 0} |n_1,n_2,\cdots,n_k\rangle\langle n_1,n_2,\cdots,n_k| = I. \quad (8)$$

- Of course, the probability interpretation only requires the matrix element p is non-negative. Here we make the stronger conjecture that the BMN two-point functions with three string modes are always non-negative separately at each genus.

- This seemingly simple equation (7) has eluded research on the topic for many years, as the physical picture of string dynamics in the infinitely curved pp-wave background turns out to be drastically different from those familiar in flat spacetime or AdS space with large radius.

- In particular, there is no finite physical process of multiple particles or strings scattering to and from asymptotic region of spacetime, as the higher point functions always vanish in the strict $N \sim \infty$ limit. (Of course, they are still very useful, as in the 1st, 2nd entries, since infinitely many of them may combine to make a finite contribution.)

- Instead, the tensionless string directly jumps from one excited state to another through a quantum unitary transition, much as in a S-matrix where the incoming and outgoing states have the same energy.

- Since the only finite BMN two-point functions are always real and symmetric, unitarity of string interactions rules out the naive possibility that they are directly identified with quantum transition amplitudes on the string theory side Huang:2019. So we arrive at the otherwise seemingly most natural conjecture (7).

Torus Two-point Functions

- The formula with k string modes

$$
\begin{aligned}
&\langle \bar{O}^J_{(m_1,m_2,\cdots,m_k)} O^J_{(n_1,n_2,\cdots,n_k)} \rangle_{\text{torus}} \\
&= \frac{g^2}{4}\int_0^1 dx_1 dx_2 dx_3 dx_4 \delta(x_1+x_2+x_3+x_4-1)\times \\
&\prod_{i=1}^{k}\Big(\int_0^{x_1} + e^{2\pi i n_i(x_3+x_4)}\int_{x_1}^{x_1+x_2} + e^{2\pi i n_i(x_4-x_2)}\int_{x_1+x_2}^{1-x_4} + e^{-2\pi i n_i(x_2+x_3)}\int_{1-x_4}^{1}\Big) dy_i e^{2\pi i(n_i-m_i)y_i} \\
&= g^2\int_0^1 dx_1 dx_2 dx_3 dx_4 \delta(x_1+x_2+x_3+x_4-1)\int_0^{x_1} dy_k e^{2\pi i(n_k-m_k)y_k}\times \\
&\prod_{i=1}^{k-1}\Big(\int_0^{x_1} + e^{2\pi i n_i(x_3+x_4)}\int_{x_1}^{x_1+x_2} + e^{2\pi i n_i(x_4-x_2)}\int_{x_1+x_2}^{1-x_4} + e^{-2\pi i n_i(x_2+x_3)}\int_{1-x_4}^{1}\Big) dy_i e^{2\pi i(n_i-m_i)y_i}.
\end{aligned}
$$

- Reality: We take the complex conjugate in the first formula, and change the integration variables $y_i \to 1-y_i, i=1,2,\cdots,k$ and $x_1,x_2,x_3,x_4 \to x_4,x_3,x_2,x_1$. After a simple calculation, also using the closed string level matching condition, one can check the formula remains the same. So the torus two-point function is purely real and symmetric.

Two String Modes

$$\langle \bar{O}^J_{(m_1,m_2)} O^J_{(n_1,n_2)} \rangle_{\text{torus}}$$
$$= \frac{g^2}{4} \int_0^1 dx_1 dx_2 dx_3 dx_4 \delta(x_1+x_2+x_3+x_4-1) \times$$
$$\prod_{i=1}^{2} \Big(\int_0^{x_1} + e^{2\pi i n_i (x_3+x_4)} \int_{x_1}^{x_1+x_2} + e^{2\pi i n_i (x_4-x_2)} \int_{x_1+x_2}^{1-x_4} + e^{-2\pi i n_i (x_2+x_3)} \int_{1-x_4}^{1} \Big) dy_i e^{2\pi i (n_i - m_i) y_i}.$$

Since $m_1+m_2 = n_1+n_2 = 0$. This is apparently non-negative.

- Higher genus: This works similarly.
  Two string modes: The non-negativity is easily proven.
  Any string modes: Flipping the diagrams vertically and horizontally, we can show the formula is always real and symmetric.

Some Standard Integrals

- We will use some standard integrals, which is defined by

$$I(u_1, u_2, \cdots, u_r) \equiv \int_0^1 dx_1 \cdots dx_r \delta(x_1 + \cdots + x_r - 1) e^{2\pi i (u_1 x_1 + \cdots u_r x_r)}.$$

It is clear that the integral is unchanged if we add an integer to all the arguments. If some of the u_i's are identical, one uses the following notation

$$I_{(a_1, \cdots, a_r)}(u_1, u_2, \cdots, u_r) \equiv I(u_1, \cdots, u_1, u_2, \cdots, u_2, \cdots, u_r, \cdots, u_r),$$

where a_i's are integers representing the numbers of the u_i's in the right hand side, and for $a_i = 0$ we can just eliminate the corresponding argument.

- The integral can be calculated by some recursion relations. A useful special case is when all arguments are degenerate at an integer

$$I_{n+1}(0) = \int_0^1 dx_1 \cdots dx_{n+1} \delta(x_1 + \cdots + x_{n+1} - 1) = \frac{1}{n!},$$

which is simply the volume of the standard n-dimensional simplex

Three String Modes

- The BMN two-point functions can be calculated in terms of the standard integrals.
  Generic case: no degeneracy of string mode numbers in the standard integrals, i.e. none of $m_i, n_i, m_i \pm n_j$'s is zero.

- The genus one formula for the generic case

$$\langle \bar{O}^J_{(m_1,m_2,m_3)} O^J_{(n_1,n_2,n_3)} \rangle_{\text{torus}} = \frac{g^2}{32\pi^4} \frac{\sum_{i=1}^3 (m_i - n_i)^2}{\prod_{i=1}^3 (m_i - n_i)^2}, \qquad (9)$$

  which is of course manifestly positive.

- We check also the numerous degenerate cases where some of $m_i, n_i, m_i \pm n_j$'s vanishes. Most cases also have manifestly non-negative results. However, there are several somewhat complicated degenerate cases. We perform a careful analysis to provide a complete proof of non-negativity at genus one for any mode numbers.

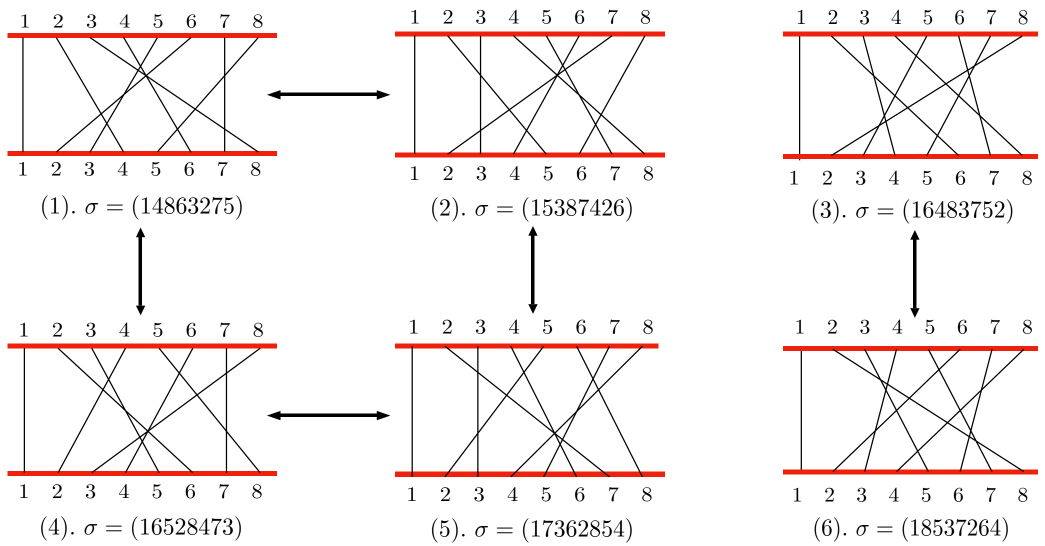
- Three string modes, Genus two: there are 21 diagrams. The calculations are quite complicated. After some very lengthy calculations, we finally obtain the result in terms of standard integrals, which is organized into four parts and too long to write down here.

- It is important to go through the laborious calculations, in order to ensure the previously observed non-negativity at genus one is not just a lucky coincidence, but more likely a manifestation of the deep mathematical structures of the underlying holographic duality.

- These 21 permutations are divided into 4 groups according to cyclicity

$$\begin{aligned}
1.\ & (14732865), (17548362), (18643725), (14875326), \\
& (15837642), (18472653), (15428736), (17625843), \\
2.\ & (15387426), (15842763), (16528473), (17362854), \\
& (17438625), (14863275), (16483752), (18537264), \\
3.\ & (14325876), (14765832), (18365472), (18725436), \\
4.\ & (16385274).
\end{aligned}$$

- Symmetries: an example



- We check numerically for all cases with mode numbers $\max(|m_i|, |n_i|) \leq 30$. Furthermore, we compute the result for more than a million random sets of mode numbers in larger range. In all tests we have not found any negative result.

- For the case of generic mode numbers with large absolute value, the dominant term can be calculated. It is actually proportional to the genus one formula and is manifestly positive.

Four String Modes

- We fixe the 4th string mode in the first segment, namely $0 < y_4 < x_1$. It turns out there are 20 cases where we can put the positions of remaining string modes $y_{1,2,3}$, up to some permutation symmetries. The 20 cases of integrals can be organized into 10 types of integrals

$$\begin{aligned}
&\langle \bar{O}^J_{(m_1,m_2,m_3,m_4)} O^J_{(n_1,n_2,n_3,n_4)} \rangle_{\text{torus}} \\
&= g^2 \sum_{(i,j,k,l)} [I_{(1,1,1,5)}(n_i - m_i, n_i - m_i + n_j - m_j, -n_k + m_k, 0) \\
&+ I_{(2,2,2,1,1)}(-m_i, -n_i, 0, -m_i + n_j - m_j, -n_i - n_k + m_k) \\
&+ I_{(2,2,2,1,1)}(m_i, n_i, 0, m_i - n_j + m_j, n_i + n_k - m_k) \\
&+ I_{(2,2,2,1,1)}(m_i + m_j, n_i + n_j, 0, m_i + n_j, -m_k - n_l) \\
&+ I_{(2,2,1,1,1,1)}(n_i - m_i, 0, n_i, -m_i, -n_j + m_j, n_i - m_i + n_k - m_k) \\
&+ I(m_i, n_i, -m_j, -n_j, 0, m_i - n_j, n_i - m_j, n_i - m_j + n_k - m_k) \\
&+ I(m_i, n_i, -m_j, -n_j, -m_j - n_j, m_i - n_j, n_i - m_j, m_i + n_i + m_k + n_l) \\
&+ I(m_i, n_i, -m_j, -n_j, m_i + n_i, m_i - n_j, n_i - m_j, m_i + n_i + m_k + n_l)] \\
&+ g^2 \sum_{(i,j)\leftrightarrow(k,l)} I_{(2,2,1,1,1,1)}(n_i + n_j - m_i - m_j, 0, n_i + n_j, -m_i - m_j, n_i - m_i, -n_k + m_k) \\
&+ g^2 \sum_{(i,j,k)} I(m_i, n_i, -m_j, -n_j, m_i + n_i + m_k, m_i + n_i + n_k, m_i + m_k - n_j, n_i + n_k - m_j).
\end{aligned}$$

- Some tests (similar to the case three string modes, genus two):
When two modes $m_4 = n_4 = 0$, this reduced to the case of three string modes.
When $m_i = 0$ and $n_i \neq 0$, then the result identically vanishes, consistent with the conservation of discrete momentum in the transverse direction.

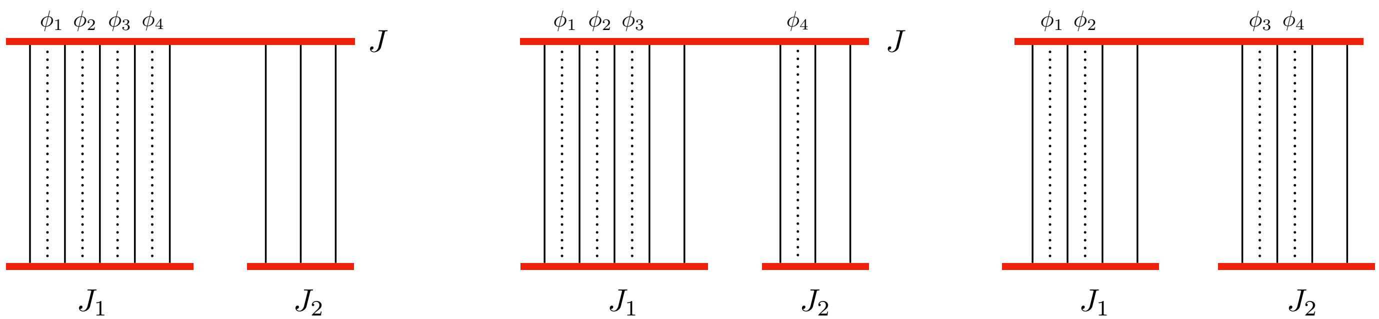
- Computing the results for some random mode numbers, we find that the result can be either positive or negative. Some empirical observations

1. If two pairs of mode numbers are the same, e.g. $m_i = n_i, i = 1, 2$, then the torus two-point functions are most likely positive. If all mode numbers are the same, i.e. $m_i = n_i, i = 1, 2, 3, 4$, then we have not found an example of negative torus two-point function.

2. For $m_i \neq n_i, i = 1, 2, 3, 4$, the sign of torus two-point function is most likely the same as $\prod_{i=1}^{4}(m_i - n_i)$.

- On the other hand, the comparison with Green-Schwarz cubic string vertex (i.e. the 1st entry) is seen to be straightforwardly applied to any hypothetical number of string modes, not even restricted by the eight dimensions of transverse directions in the pp-wave background.
  We also confirm that the factorization formulas (i.e. the 2nd entry) are still valid for the case of four string modes.

- The free planar three-point functions



$< \bar{O}^J_{(m_1,m_2,m_3,m_4)} O^{xJ}_{(n_1,n_2,n_3,n_4)} O^{(1-x)J} >$ $\quad$ $< \bar{O}^J_{(m_1,m_2,m_3,m_4)} O^{xJ}_{(n_1,n_2,n_3)_{(1,2,3)}} O^{(1-x)J}_{(0)_4} >$ $\quad$ $< \bar{O}^J_{(m_1,m_2,m_3,m_4)} O^{xJ}_{(-n_1,n_1)_{(1,2)}} O^{(1-x)J}_{(-n_2,n_2)_{(3,4)}} >$

Higher point correlators, including the planar three-point functions, actually always vanish in the strict BMN limit, and are now perceived by us as a kind of virtual processes.

- The Green-Schwarz cubic string vertex were computed in the pp-wave background Spradlin:2002 and becomes much simplified in the infinite curvature limit Huang:2002.

- In the study of superstring field theories in flat space, besides the cubic vertex, there are other important physical quantities, such as the pref-actor and the higher order contact interactions. With our specialization to the infinite curvature limit, the tensionless strings do not have an effective action description. So in our case it seems that at tree level, the cubic vertex is the only remaining relevant finite physical object to consider.

- The (1st) entry of the pp-wave holographic dictionary

$$\frac{\langle \bar{O}_3 O_1 O_2 \rangle}{\langle \bar{O}^J O^{xJ} O^{(1-x)J} \rangle} = \frac{\langle 1|\langle 2|\langle 3|V\rangle}{\langle 0|V\rangle}, \tag{10}$$

  Of course, as we mentioned, the planar three-point functions are vanishing in the BMN limit $J \sim \infty$ and regarded as virtual processes, but their ratios with the vacuum correlator are finite and meaningfully related to the cubic string vertex.

- The Factorization Formulas (2nd entry) at genus one

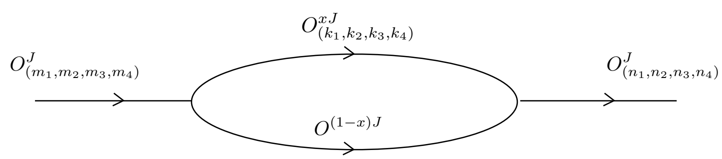
$$\langle \bar{O}^J_{(m_1,m_2,m_3,m_4)} O^J_{(n_1,n_2,n_3,n_4)} \rangle_{\text{torus}} = g^2 \sum_{i=1}^{10} I_i$$

$$= \frac{J}{2}[\int_0^1 dx \sum_{\sum_i k_i=0} \langle \bar{O}^J_{m_1,m_2,m_3,m_4} O^{J_1}_{k_1,k_2,k_3,k_4} O^{J_2} \rangle \langle \bar{O}^{J_1}_{k_1,k_2,k_3,k_4} \bar{O}^{J_2} O^J_{n_1,n_2,n_3,n_4} \rangle$$

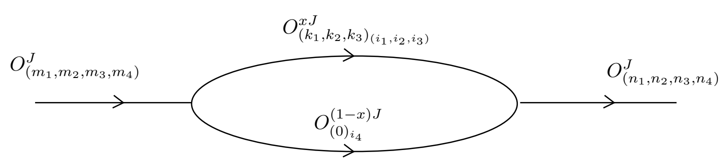
$$+ \int_0^1 dx \sum_{i_4=1}^{4} \sum_{\sum_i k_i=0} \langle \bar{O}^J_{m_{i_1},m_{i_2},m_{i_3},m_{i_4}} O^{J_1}_{k_1,k_2,k_3} O^{J_2}_0 \rangle \langle \bar{O}^{J_1}_{k_1,k_2,k_3} \bar{O}^{J_2}_0 O^J_{n_{i_1},n_{i_2},n_{i_3},n_{i_4}} \rangle$$

$$+ \int_0^1 dx \sum_{i_2=2}^{4} \sum_{k,l=-\infty}^{+\infty} \langle \bar{O}^J_{m_1,m_{i_2},m_{i_3},m_{i_4}} O^{J_1}_{-k,k} O^{J_2}_{-l,l} \rangle \langle \bar{O}^{J_1}_{-k,k} \bar{O}^{J_2}_{-l,l} O^J_{n_1,n_{i_2},n_{i_3},n_{i_4}} \rangle].$$

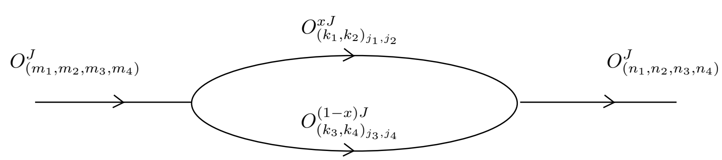
S_1



S_2



S_3



Five String Modes: generic case

- For the generic situation with no degeneracy, we can first perform the y_i's integrals

$$\begin{aligned}
&\langle \bar{O}^J_{(m_1,m_2,\cdots,m_5)} O^J_{(n_1,n_2,\cdots,n_5)}\rangle_{\text{torus}} \\
&= \frac{g^2}{(2\pi i)^5 \prod_{i=1}^5 (n_i - m_i)} \int_0^1 dx_1 dx_2 dx_3 dx_4 \delta(x_1+x_2+x_3+x_4-1)(e^{2\pi i(n_5-m_5)x_1}-1) \\
&\times \prod_{i=1}^4 [e^{2\pi i(n_i-m_i)x_1} - 1 + e^{-2\pi i m_i(x_1+x_2)} - e^{2\pi i(-m_i x_1 - n_i x_2)} + e^{2\pi i(-n_i x_2 + m_i x_4)} \\
&- e^{2\pi i[(n_i-m_i)x_1 - m_i x_2 + n_i x_4]} + e^{-2\pi i n_2(x_2+x_3)} - e^{2\pi i(n_2 x_1 + m_2 x_4)}].
\end{aligned}$$

  So this calculation becomes some standard 4-dimensional integrals.

- Using the reality of the integral, it turns out the calculations are especially simple

$$\begin{aligned}
&\langle \bar{O}^J_{(m_1,m_2,\cdots,m_5)} O^J_{(n_1,n_2,\cdots,n_5)}\rangle_{\text{torus}} \\
&= \frac{g^2}{(2\pi)^6 \prod_{i=1}^5 (n_i - m_i)} [\sum_{i=1}^5 \frac{1}{n_i - m_i} - \sum_{i=1}^4 \sum_{j=i+1}^5 \frac{1}{n_i - m_i + n_j - m_j}].
\end{aligned}$$

  We can compute the result for some random mode numbers, and find that it can be either positive or negative.

- Some issues with multiple string modes in the same transverse direction. Naively, the corresponding BMN operators can be similarly constructed, using the same scalar field (or covariant derivative) going through the string of Z's with multiple sums with phases, with possibly a different normalization.

- However, this brings a subtle issue. We recall that the chiral primary operators with lowest dimension in a short multiplet of $\mathcal{N} = 4$ super-Yang-Mills theory are constructed by the 6 real scalars in the $SO(6)$ symmetric traceless representation. They are BPS operators whose conformal dimensions are protected by supersymmetry.

- When a real scalar appears multiple times, an operator may no longer be chiral primary. For example, the operator $\mathrm{Tr}((\phi^I)^2)$, known as the Konishi operator, is not a chiral primary operator, since it is not traceless in the $SO(6)$.
  On the other hand, the BMN vacuum operator $\mathrm{Tr}(Z^J)$ is a chiral primary operator since a power of the complex scalar Z is automatically traceless in the $SO(6)$.

- In the original calculations of planar anomalous conformal dimensions of the BMN operator $O^J_{-m,m}$, one used the fact that for $m = 0$, the operator $O^J_{0,0}$ is a chiral primary operator whose conformal dimension is not corrected by gauge interactions.

- So one only needs to compute the mode number m-dependent part which is perturbative in an effective gauge coupling constant $\lambda' \equiv \frac{g^2_{YM}N}{J^2}$, a small parameter in the BMN limit.

- In this sense the BMN operators of distinct scalar field insertions with non-zero modes are "near BPS" operators. If we put two identical real scalars into the string of Z's, the zero mode operator, namely $O^J_{[0,0]}$, is no longer a chiral primary operator. There may be large (field theory) quantum corrections to the m-independent part of its conformal dimension. So in this case the calculations of planar conformal dimension is no longer reliable. We are not aware a simple natural fix which also matches the expectations from the string theory side.

- In any case, we may hope by restricting ourselves to free gauge theory, this issue with large quantum gauge corrections does not cause problems.

- We check that the mathematical structures in the factorization formula (2nd entry) and comparison with cubic string vertex (1st entry) are robust and remain valid in this situation as we stay in free gauge theory.

- However, the proposed probability interpretation (3rd entry) again seems rather fragile and further breaks down in the case of three string modes because of a problem with normalization, though it still holds up in the case of two string modes due to the decoupling of the zero mode with non-zero modes.

Conclusion

- The $SO(8)$ rotational symmetry of the transverse directions in the pp-wave background (1) is broken by the Ramond-Ramond flux into $SO(4)\times SO(4)$, where the bosonic string modes are described differently by covariant derivatives and scalar field insertions in the dual CFT.

- As such, it is reasonable to expect our proposed entries of pp-wave holographic dictionary to face some challenges with more than four distinct string modes as the infinite Ramond-Ramond flux in our setting shall separate the two types of string modes.

- However, it is rather surprising that even for the case of four string modes, the torus two-point function can be negative, so the probability interpretation (i.e. the 3rd entry) may no longer valid.

- On the other hand, the comparison with cubic string vertex (i.e. the 1st entry) is seen to be straightforwardly applied to any hypothetical number of string modes, not even restricted by the eight dimensions of transverse directions in the pp-wave background, while we confirm that the factorization formulas (i.e. the 2nd entry) are still valid for the case of four string modes.
  Question: Does the factorization formulas (i.e. the 2nd entry) works for more than four modes, even any hypothetical number of modes?

- Of course, since the two-point function is always real and symmetric, the arguments in Huang:2019 are still valid that it can not be naively identified with a quantum transition amplitude on the string theory side, which would then violate fundamental principle of unitarity. It would be desirable to better understand the physical meaning of the two-point function on the string theory side of the correspondence in this situation.

- As mentioned in Huang:2019, the probability interpretation of two-point function implies the genus expansion is convergent. In this sense, the holographic higher genus calculations are not asymptotic perturbative expansions as familiar in most examples of quantum theories.

- If no new non-perturbative effect is discovered in the future, then perhaps we have luckily found a rare example of perturbatively complete string theory.

- Thus, if our proposal of the entry of pp-wave holographic dictionary (7) is correct, to our knowledge, it would not only provide first examples of systematic calculations of (the norms of) the higher genus critical superstring amplitudes, but may also in principle gives exact complete results for any string coupling, due to the convergence of genus expansion.

- There is no technical obstruction for our calculations on the free gauge theory side at any genus, however the available tools on the string theory side are very limited. Much progress for the calculations of higher genus critical superstring amplitudes focused on using the RNS formalism in flat space, and is already quite difficult at genus two.

- Our conjecture (7) gives the norms of certain critical superstring amplitudes including all genus contributions. Of course, the string amplitudes are complex, and consist of the norms and phase angles. As discussed in Huang:2019, unitarity can in principle determine a large part of the phase angles, but not completely.

- Our studies thus provide a long term motivation for future research to develop techniques that can deal with string theory on highly curved background, with flux, and including highly excited stringy states, for the purposes of a direct verification of the conjecture (7) as well as the complete determination of string amplitudes including the phase angles.
  For the moment, our verifications of non-negativity on the gauge theory side provide indirect non-trivial evidence of the conjecture (7).

- Now that we are more confident in the validity of the conjectured non-negativity for three string modes, it would be certainly better to search for a universal analytic proof, at genus two and further at any higher genus.

- Although the integrals are completely elementary, it seems a complete proof may require some advanced mathematical techniques. Such a proof may elucidate the surprising entry (7) of the pp-wave holographic dictionary.

- It is well known that string theory has a Hagedorn temperature inverse proportional to the string length, which may be interpreted as a maximal temperature or a temperature of phase transition.

- In our physical setting, the tensionless strings have effectively infinite length, so the Hagedorn temperature is zero. This is not necessarily a problem, since for example an extremal black hole emits no Hawing radiation, has zero temperature, but can still have finite event horizon and entropy. It would be interesting to see whether we can further study some interesting thermodynamics under such extreme conditions.

- It is widely believed that strings may not be the right fundamental degrees of freedom to formulate the still mysterious M-theory, since there are non-perturbative objects like D-branes and M-branes.

- On the contrary, if our conjecture (7) is correct, it seems that in our very special setting with infinite spacetime curvature and Ramond-Ramond flux, the tensionless closed strings do provide the proper complete degrees of freedom for physics at any coupling constant.

- In other words, we conjecture that in this case, string theory is really a theory of just "strings" .

- Although we focus on a highly unrealistic special situation, our studies may provide some insights for the non-perturbative formulation of string/M-theory on general backgrounds.

Thank You