Twisted Compactification of 6d SCFTs

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KL,June Nahmgoong ('20); **Zhihao Duan**, KL, June Nahmgoong and **Xin Wang**, ('21); KL, **Kaiwen Sun** on 5d to appear; KL, **Kaiwen Sun**, **Xin Wang** on 6d to appear.

Goal

* define twisted circle compactification of 6d theory to 5d

* set up formalisms for obtaining their partition function on $T^2 \times \mathbb{R}^4_{\epsilon_1,\epsilon_2}$

Blow-up formula and modular bootstrap

check and help each other to find the fuller answer

* take the Cardy limit and relate the partition function to black hole entropy

Outline

- Introduction
- twisting
- S-dual
- Blow up formula on $T^2 \times \mathbb{R}^4_{\epsilon_{1,2}}$
- Elliptic genus for the partition function on $T^2 \times \mathbb{R}^4_{\epsilon_{1,2}}$
- Cardy Limit
- Concluding remarks

Introduction

6d (2,0) Superconformal Field Theories

- C^2/Γ_G in type IIB with G=ADE classification, Non-Lagrangian, Witten'95
- D3 wrapping two-cycles= selfdual string for self-dual tensor: $H=dB={}^*H$, Φ_A, Ψ_α
- SO(5) R-symmetry $H=dB={}^*H$, Φ_A, Ψ: (**0**, **5**, **4**)
- A_N type: N+1 M5 branes + M2 branes between them Strominger'95
- D_N type: N M5 branes on $OM5^-$:
- N-cubic degrees of freedom

6d (1,0) Superconformal Field Theories

- UV complete QFT in 6d with (1,0) supersymmetry
- $SU(2)_R$ symmetry
- Tensor multiplet: $H=dB={}^*H$, Φ, Ψ (1,0)
- Vector multiplet: $F=dA$, $\lambda_{(0,1)}$
- Matter mutiplet: q_A, $\psi_{(1,0)}$.
- Coupling: $\Phi \, \mathrm{tr}\, F^2 + B \wedge \mathrm{tr}\, F \wedge F^*$
- Anomaly Cancellation: Gauge Anomalies & Green-Schwarz mechanism
- Classification: F-theory on singular elliptically fibered Calabi-Yau 3-folds

Heckman,Morrison,Rudelius,Vafa ('15),Heckman,Morrison,Vafa('13)DelZotto,Heckman,Morrison('17), Bhardwaji ('15,19)

5d N=2 Super Yang-Mills Theories

- Circle compactification of 6d (2,0) A_N theory on a circle: $x_5 \sim x_5 + 2\pi R_6$
- N+1 M5 brane wrapping M-circle: N+1 D4 branes
- 5d N=2 super Yang-Mills theory of gauge group SU(N+1)
- D0 branes on D4 branes: Yang-Mills Instanton
- Instantons=Kaluza-Klein modes

Seiberg'97

- $$\frac{8\pi^2}{g_5^2} = \frac{1}{R_6}$$

4d N=4 SU(N) gauge theories

- Additional Compactification to a circle: $x_4 \sim x_4+2\pi R_5$
- 4d Yang-Mills coupling constant: $\frac{1}{\alpha_4} = \frac{4\pi}{g_4^2} = \frac{8\pi^2 R_5}{g_5^2} = \frac{R_5}{R_6}$
- Change of the ordering of complactification $R_5 \leftrightarrow R_6$: S-duality: $\alpha_4 \leftrightarrow \frac{1}{\alpha_4}$
- Shifted compactification: $x_4 \sim x_4+2\pi R_5$, $x_5 \sim x_5+\theta R_6$
  - Complex Coupling: $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g_4^2}$

5d N=2 SYM with non-simple-laced group?

- B_r=SO(2r+1), C_r=Sp(r)$_0$, C'$_r$=Sp(r)$_\pi$, F_4, G_2

Hori 9805141,Gimon 9806226, Hanany 0003025

- r D4 branes on $O4^-$, $\widetilde{O4}^-$, $O4^+$, $\widetilde{O4}^+$: SO(2r), SO(2r+1), Sp(r)$_0$, Sp(r)$_\pi$
- D4 brane charge: $O4^- : -\frac{1}{2},\ \widetilde{O4}^- : 0,\ O4^+ : \frac{1}{2},\ \widetilde{O4}^+ : \frac{1}{2}$
- D0 branes on D4 branes: Yang-Mills instanton

Seiberg'97

- How to get it from 6d (2,0) theories?

twisting

S-duality=exchange of order of compactification

6d (2,0) G_0-type SCFT

on S^1 of R_6

twisting — on $\tilde{S}^1$ of $\tilde{R}_6 = n_G R_5$

-

5d MSYM with G_0

5d MSYM with $G^\vee$

twisting — on S^1 of R_5

on $\tilde{S}^1$ of $\tilde{R}_5 = R_6$

S-dual

4d MSYM with $G_0^{(n)}$ ⟷ 4d MSYM with $G^{\vee(1)}$

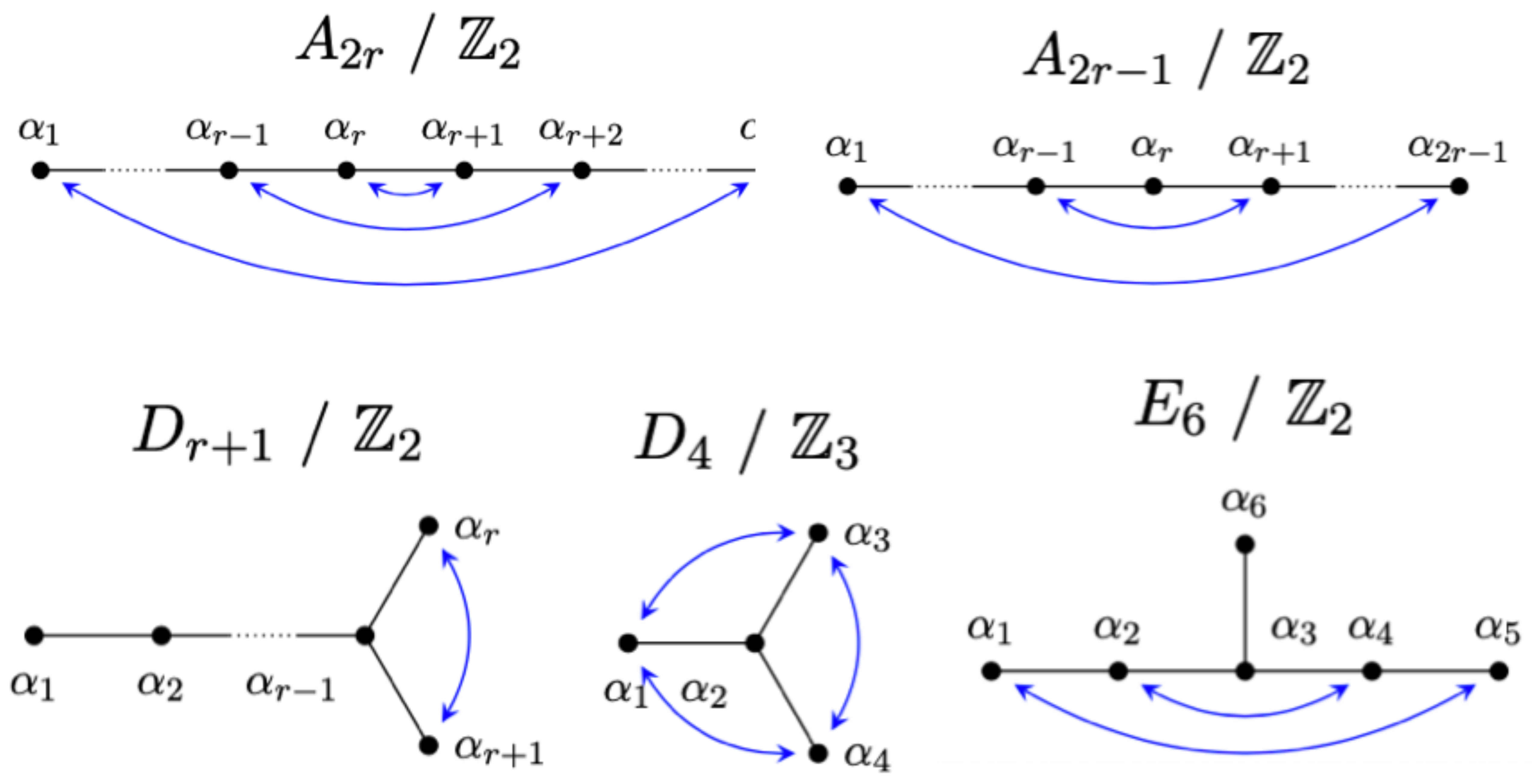
Tachikawa ('11)

5d G_0=ADE theories on S^1 of R_5 with twist

- Twisting from 5d to 4d with outer automorphism σ:

$$\phi(x_5 + 2\pi R_5) = \sigma(\phi(x_5))$$

- A_{2r}=SU(2r+1), A_{2r-1}=SU(2r), D_{r+1}=SO(2r+2), D_4=SO(8), E_6



5d G$_0$=ADE theories with twisted on S^1 of R_5

- outer automorphism of Dynkin diagram: Z_2 or Z_3
- fractional KK momentum $p_5 = \frac{\mathbb{N}}{n_G R_5}, \mathbb{N} \in \mathbb{Z}$
- $n_G = 2{,}3{,}4$

SU(3)-> SU(2): 8=3+2$_{1/4}$+2$_{3/4}$+1$_{1/2}$

$$
\begin{aligned}
A_{2r}^{(2)} &: \textbf{adj of } A_{2r} \quad n_G = 4 && \longrightarrow && \text{long}_k \oplus \text{short}_{\frac{k}{2}} \oplus \text{special}_{k\pm\frac{1}{4}} \oplus 1_{k+\frac{1}{2}} \quad \text{of } C_r' \\
A_{2r-1}^{(2)} &: \textbf{adj of } A_{2r-1} && \longrightarrow && \text{long}_k \oplus \text{short}_{\frac{k}{2}} \quad \text{of } C_r \\
D_{r+1}^{(2)} &: \textbf{adj of } D_{r+1} && \longrightarrow && \text{long}_k \oplus \text{short}_{\frac{k}{2}} \quad \text{of } B_r \\
E_6^{(2)} &: \textbf{adj of } E_6 && \longrightarrow && \text{long}_k \oplus \text{short}_{\frac{k}{2}} \quad \text{of } F_4 \\
D_4^{(3)} &: \textbf{adj of } D_4 && \longrightarrow && \text{long}_k \oplus \text{short}_{\frac{k}{3}} \quad \text{of } G_2 \ ,
\end{aligned}
$$

5d G_0=ADE theories on S^1 of R_5 with twist

- twisted affine algebra $G_0^{(n)}$ & 4d theory in low energy

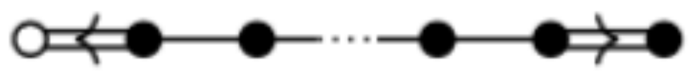
$A_{2r}^{(2)}$ (4d C_r')

$\widetilde{\mathrm{O3}}^-$ $\widetilde{\mathrm{O3}}^+$

$A_{2r-1}^{(2)}$ (4d C_r)

$\mathrm{O3}^-$ $\mathrm{O3}^+$

$D_{r+1}^{(2)}$ (4d B_r) .



$\widetilde{\mathrm{O3}}^-$ $\widetilde{\mathrm{O3}}^-$

$D_4^{(3)}$ (4d G_2)

$E_6^{(2)}$ (4d F_4)

5d G₀=ADE theories with twisted on S^1 of R_5

- $n_G = 4$ for $A_{2r}^{(2)}$
- fractional KK momentum $p_5 = \dfrac{\mathbb{N}}{4R_5}$, $\mathbb{N} \in \mathbb{Z} \geq 0$
- Imbedding Sp(r)=USp(2r) in A_{2r}=SU(2r+1) is of order 4.

$G^{(n)}$	4d	Long	Short	Special
$A_{2r}^{(2)}$	C'_r	$\pm\sqrt{2}e_a$	$\frac{1}{\sqrt{2}}(\pm e_a \pm e_b)$	$\pm\frac{1}{\sqrt{2}}e_a$

$$A_{2r}^{(2)} : \ \textbf{adj of } A_{2r} \quad n_G = 4 \quad \longrightarrow \qquad \text{long}_k \oplus \text{short}_{\underline{k}} \oplus \text{special}_{k\pm\frac{1}{4}} \oplus 1_{k+\frac{1}{2}} \ \text{ of } C'_r$$

affine root of
KK momentum ¼

S-dual

4d orientifold

- Under S: exchange W-bosons and magnetic monopoles: $\alpha \leftrightarrow \alpha^\vee = \dfrac{2\alpha}{\alpha^2}$

- r D3 on $O3^-$, $\widetilde{O3}^-$, $O3^+$, $\widetilde{O3}^+$: D_r, B_r, C_r, C_r'

  D_r: self-dual, $B_r \leftrightarrow C_r$, C_r' : self-dual

- Under T: $O3^+ \leftrightarrow \widetilde{O3}^+$.

The range [0,4π] of θ for Sp(r) is divided into [0,2π] for C_r=Sp(r), and [2π,4π] for C_r'=Sp(r)'.

Magnetic dual of the long root is dyonic and need a bound state of two dyons to make it charge neutral.

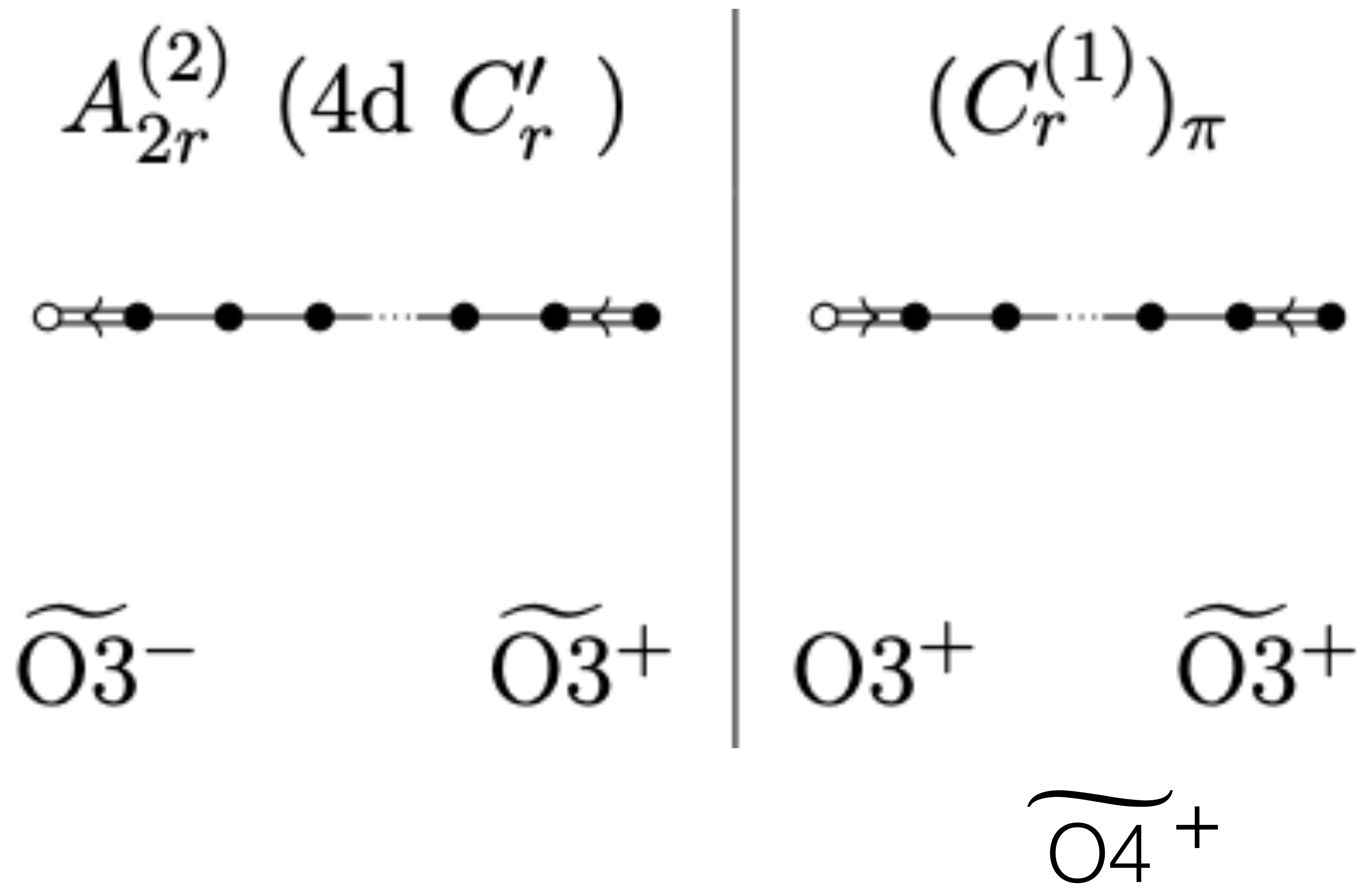
Langlands dual $G^{\vee(1)}$ of 4d G in $G^{(n)}$

- S-duality: reverse the arrow in the twisted affine Dynkin diagram except $A_{2r}^{(2)}$ case.

-

$G^{(n)}$(4d G')	$G^{\vee(1)}$	n_G
$A_{2r}^{(2)}$(4d C'_r)	$(C_r^{(1)})_\pi$	4
$A_{2r-1}^{(2)}$(4d C_r)	$B_r^{(1)}$	2
$D_{r+1}^{(2)}$(4d B_r)	$(C_r^{(1)})_0$	2
$D_4^{(3)}$(4d G_2)	$G_2^{(1)}$	3
$E_6^{(2)}$(4d F_4)	$F_4^{(1)}$	2

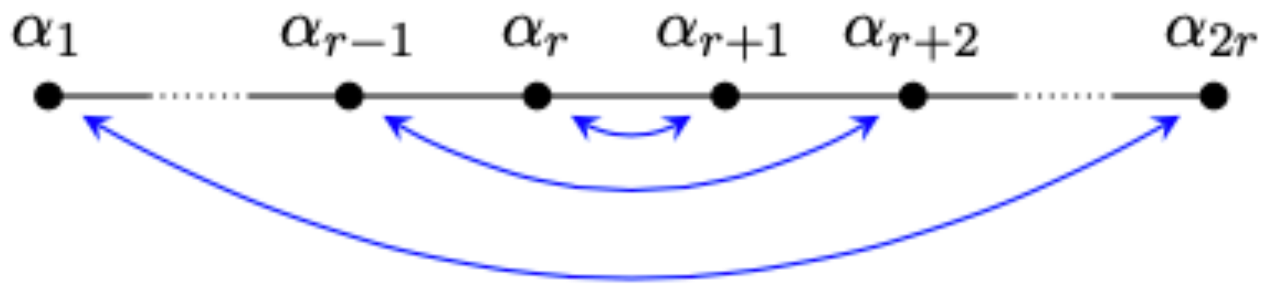
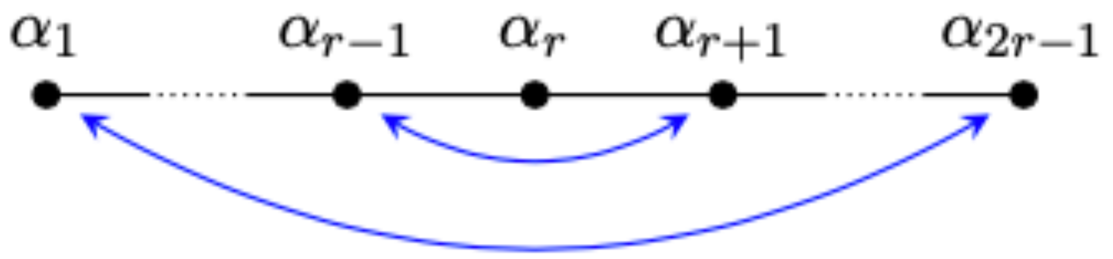
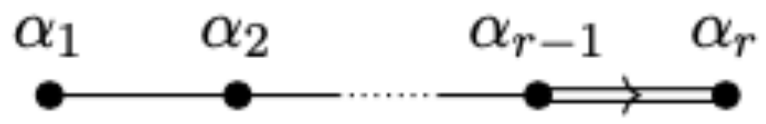
S-dual

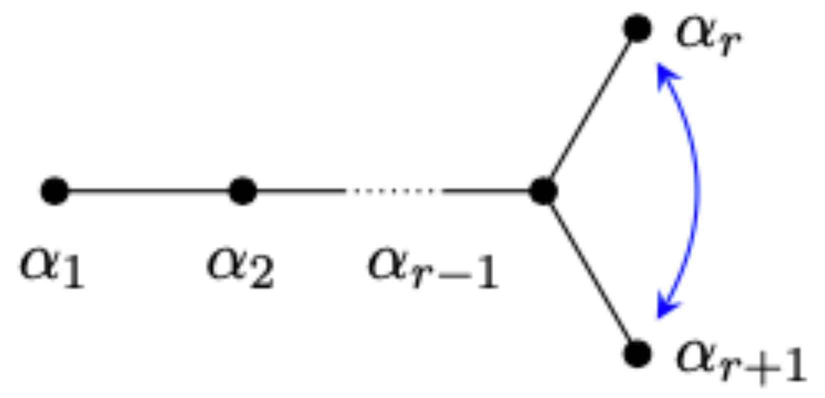
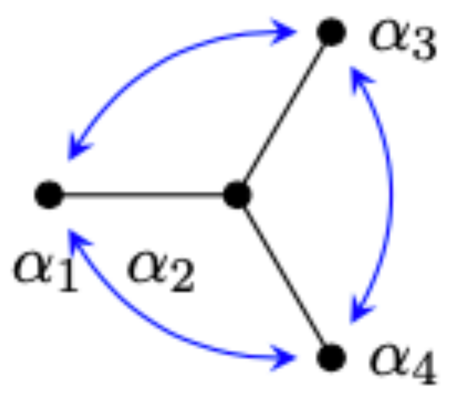
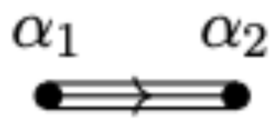
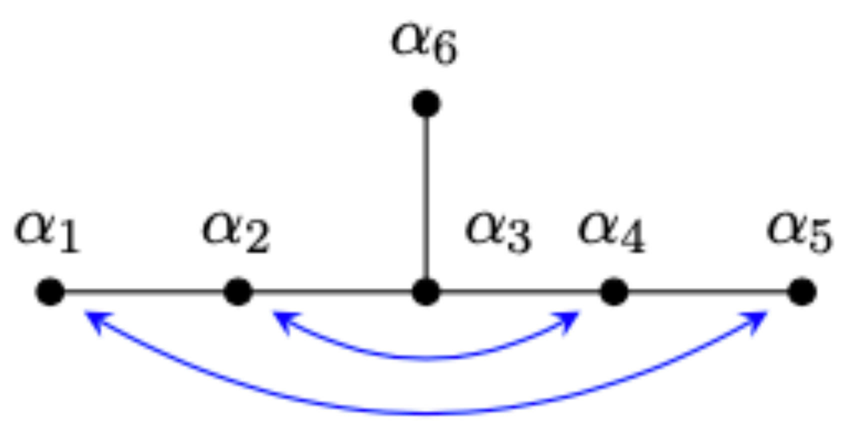
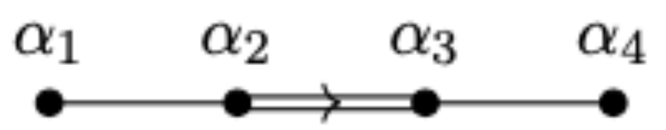


S-duality=exchange of order of compactification

- radius $R_6 = \tilde{R}_5$, radius $n_G R_5 = \tilde{R}_6$ with $n_G = 1,2,3,4$
- KK momentum $\dfrac{\mathbb{N}}{n_G R_5} = \dfrac{\mathbb{N}}{\tilde{R}_6}$, $\mathbb{N} \in \mathbb{Z}$: $\alpha^2_{\text{long}} = 2$
- $\dfrac{4\pi}{g_4^2} = \dfrac{R_5}{R_6}$, $\dfrac{1}{\alpha_{45}^{\vee}} = \dfrac{4\pi^2}{g_4^{\vee 2}} = \dfrac{\tilde{R}_5}{\tilde{R}_6} = \dfrac{R_6}{n_G R_5} = \dfrac{1}{n_G}\dfrac{g_4^2}{4\pi} = \dfrac{\alpha_{4d}}{n_G}$
- $n_G \cdot \tau \cdot \tau^{\vee} = -1$

6d (2,0) G type **4d (2,0) $G^{(n)}$ type**

G / $\mathrm{Out}(G)$	$G^{(n)}$ (4d G')	$G^{\vee(1)}$	5d $G^{\vee}$
A_{2r} / $\mathbb{Z}_2$ 	$A_{2r}^{(2)}$ (4d C_r')  $\widetilde{\mathrm{O3}}^-$ $\widetilde{\mathrm{O3}}^+$	$(C_r^{(1)})_\pi$  $\mathrm{O3}^+$ $\widetilde{\mathrm{O3}}^+$	$(C_r)_\pi$  $\widetilde{\mathrm{O4}}^+$
A_{2r-1} / $\mathbb{Z}_2$ 	$A_{2r-1}^{(2)}$ (4d C_r)  $\mathrm{O3}^-$ $\mathrm{O3}^+$	$B_r^{(1)}$  $\mathrm{O3}^-$ $\widetilde{\mathrm{O3}}^-$	B_r  $\widetilde{\mathrm{O4}}^-$

G / Out(G)	$G^{(n)}$ (4d G')	$G^{\vee(1)}$	5d $G^{\vee}$
D_{r+1} / $\mathbb{Z}_2$ 	$D_{r+1}^{(2)}$ (4d B_r)  $\widetilde{\text{O3}}^-$ $\widetilde{\text{O3}}^-$	$(C_r^{(1)})_0$  O3$^+$ O3$^+$	$(C_r)_0$  O4$^+$
D_4 / $\mathbb{Z}_3$ 	$D_4^{(3)}$ (4d G_2) 	$G_2^{(1)}$ 	G_2 
E_6 / $\mathbb{Z}_2$ 	$E_6^{(2)}$ (4d F_4) 	$F_4^{(1)}$ 	F_4 

Blow-up Formula on Partition Function on $T^2 \times \mathbb{R}^4_{\epsilon_{1,2}}$

6d (2,0) SCFTs on a circle S^1

- 5d description in the Coulomb branch: ADE gauge theory obtained without twist
  - ¼ BPS dyonic instantons with large angular momentum and degeneracy
- Nekrasov Partition function on $R^4_{\epsilon_{1,2}} \times S^1$ with $q = e^{2\pi i\tau} = e^{-\beta}$
  - $Z(\tau, \mathbf{v}, \epsilon_{1,2}, m) = \mathrm{Tr}(-1)^F e^{-\beta\{Q,Q^\dagger\}} e^{-\epsilon_1(J_1+J_R)} e^{-\epsilon_2(J_2+J_R)} e^{-mJ_m} e^{-v_A T_A}$
  - $Z = Z_{\text{pert}} Z_{\text{instanton}}$ $\quad Z_{\text{instanton}} = 1 + \sum_{k=1}^{\infty} q^k Z_k$
  - 5d N=1* SU(N) case H.C.Kim,S.Kim Koh,KL,SLee1 110.2175
  - 5d N=1* theories with orientifolds Hwang,Kim,Kim,1607.08557

Gopakumar-Vafa Invariants for 5d N=2

- Counting BPS dyonic instanton states for given electric charge, angular momenta and R-charge
- $Z = Z_{\text{pert}} \times Z_{\text{instanton}} = \text{PE}\left[\mathcal{F}_{GV}\right]$
  - $\mathcal{F}_{GV} = \sum_{j_-, j_+ = 0}^{\infty} \sum_{\beta} (-1)^{2(j_- + j_+)} f_{j_-, j_+}(q_1, q_2) N^{\beta}_{j_-, j_+} Q_\beta$
  - $q_1 = e^{-\epsilon_1},\ q_2 = e^{-\epsilon_2},\ \ Q_\beta = \{q = e^{2\pi i \tau}, e^{-\mathbf{v}}, e^{-m}\}$
  - $f_{j_\mp}(q_{1,2}) = \dfrac{\chi_{j_-}(u)\chi_{j_+}(v)}{(q_1^{\frac{1}{2}} - q_1^{-\frac{1}{2}})(q_2^{\frac{1}{2}} - q^{-\frac{1}{2}})},\ u = (q_1 q_2)^{\frac{1}{2}}, v = (q_1/q_2)^{\frac{1}{2}}$
- Gopakumar-Vafa Invariants: $N^{\beta}_{j_-, j_+} \in \mathbb{Z} \geq 0$ $\quad N^{adj}_{0,0}$: hyper, $N^{adj}_{0,\frac{1}{2}}$: vector

Blow-up formula approach for Z

- Nakajima-Yoshioka Blow-up Formula

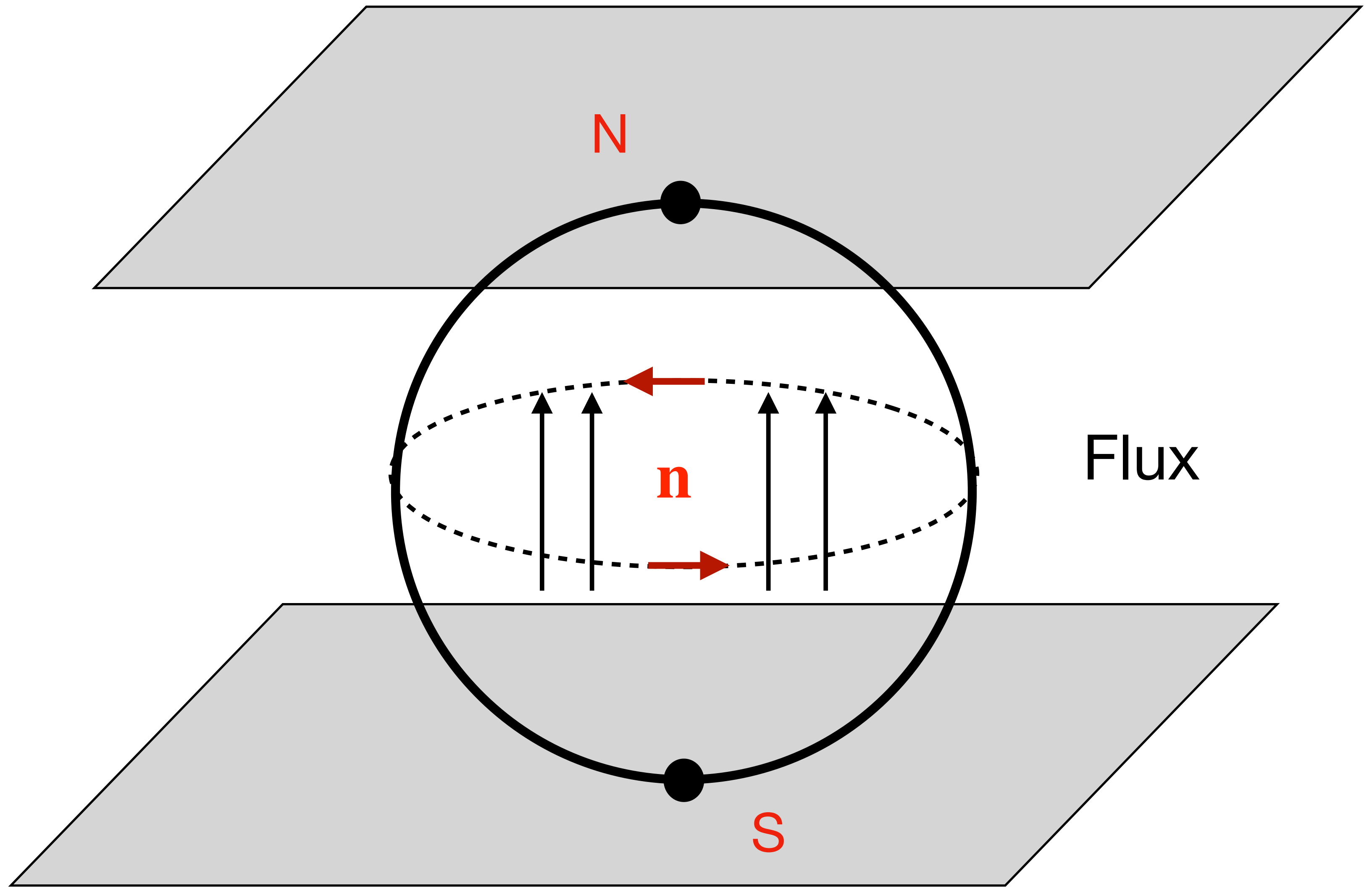
  - Extension of the partition function: $Z = Z_{\text{classical}} \times Z_{\text{1-loop}} \times Z_{\text{instanton}}$

  - $Z_{\text{classical}} = \frac{1}{\epsilon_1 \epsilon_2}\left(\mathscr{F}(\mathbf{v}) + \cdots\right)$: 1-loop prepotential+ mixed gauge/gravitaional CS term+ mixed gauge/$SU(2)_R$ CS term

- Blow up $\mathbb{C}^2$ at the origin: $\hat{\mathbb{C}}^2 = \{\mathcal{O}(-1) \to \mathbb{P}^1\}$ with $(z_0; z_1, z_2)$ such that $(z_0; z_1, z_2) \sim (\lambda^{-1} z_0; \lambda z_1, \lambda z_1)$

Nakajima and Yoshioka ('03),
Keller,Song('12),Huang,Sun,Wang('17)
Kim,Kim,Lee,KL,Song('19),
Gu,Haghighat,Sun,Wang('18)
Gu,Haghighat,Klemm,Sun,Wang('19,'20)
Kim,Kim,Kim,Lee ('21)

Blow up Formula



- Two fixed points: North and South pole

- Localize $U(1)_{\epsilon_1} \times U(1)_{\epsilon_2}$ action: $(z_0; z_1, z_2) \to (z_0; e^{\epsilon_1} z_1, e^{\epsilon_2} z_1)$ with two fixed points

- N/S pole (0;1,0)/(0;0,1): $Z_{\hat{\mathbb{C}}^2}(\mathbf{v}, \epsilon_{1,2}, q, m) = \sum_{\mathbf{n} \in Q^\vee} (-1)^{|\mathbf{n}|+|\phi|} Z^{\mathrm{N}}_{\mathbf{n}} Z^{\mathrm{S}}_{\mathbf{n}}$: locally flat $\mathbb{C}^2$

  co-root lattice

- Blow down $\mathbb{P}^1$: $Z_{\hat{\mathbb{C}}^2} = \Lambda(\epsilon_{1,2}, q, \mathbf{m}) \cdot Z_{\mathbb{C}^2}$ with the factor Λ independent of the Coulomb parameter $\mathbf{a}$.

Blow up Formula

[Nakajima and Yoshioka, 0505553]
[Huang, Sun and Wang, 1711.09884][Kim et al, 1908.11276]
[Kim, Kim, Kim and Lee, 2101.00023]

- Sum over the gauge magnetic flux $\mathbf{n}$ on two sphere with external gauge and flavor magnetic fluxes

- Partition functions:

  - $Z^{\mathrm{N}}_{\mathbf{n}} = Z_{\mathbb{C}^2}(\mathbf{v} + (\mathbf{n}+\boldsymbol{\lambda}_G)\epsilon_1, \epsilon_1, \epsilon_2 - \epsilon_1, qe^{r_b\epsilon_1}, \mathbf{m}+\boldsymbol{\lambda}_F\epsilon_1)$

  - $Z^{\mathrm{S}}_{\mathbf{n}} = Z_{\mathbb{C}^2}(\mathbf{v} + (\mathbf{n}+\boldsymbol{\lambda}_G)\epsilon_2, \epsilon_1 - \epsilon_2, \epsilon_2, qe^{r_b\epsilon_2}, \mathbf{m}+\boldsymbol{\lambda}_F\epsilon_2)$

    - Various constraints on $\boldsymbol{\lambda}_G, r_b, \boldsymbol{\lambda}_F$ $\mathbf{n} + \boldsymbol{\lambda}_G$ ← co-weight lattice

- Multiple blow up formulas for different choices of $\boldsymbol{\lambda}_G, r_b, \boldsymbol{\lambda}_F$

Blow up Formula

[Nakajima and Yoshioka, 0505553]
[Huang, Sun and Wang, 1711.09884][Kim et al, 1908.11276]
[Kim, Kim, Kim and Lee, 2101.00023]

- Multiple blow up formulas
- Different choice of flux (λ_G, λ_F) leads many different blow up formula:
  $Z_{\hat{\mathbb{C}}^2}(\mathbf{v}, \epsilon_{1,2}, q, \mathbf{m}) = \Lambda(\epsilon_{1,2}, q, \mathbf{m}) \cdot Z_{\mathbb{C}^2}(\mathbf{v}, \epsilon_{1,2}, q, \mathbf{m})$
- unity blow up formula when $\Lambda(\epsilon_{1,2}, q, \mathbf{m}) \neq 0$,
- vanishing blow up formula when $\Lambda(\epsilon_{1,2}, q, \mathbf{m}) = 0$

Blow up Formula

[Nakajima and Yoshioka, 0306198]
[Keller and Song, 1205.4722]
[Huang, Sun and Wang, 1711.09884]]
[Kim,Kim,Lee,Lee,Song, 1908.11276]
[Gu,Haghighat,Sun,Wang,1811.02577]
[Gu,Haghighat,Klemm,Sun,Wang,1911.11724, 2006.03030]
[Kim,Kim,Kim,Lee 2101.00023]

- Allowed $\lambda_G \in \Lambda_{\text{coweight}}/\Lambda_{\text{coroot}}$
- The number of non-equivalent $\lambda_G = \det(\Omega)$ with the Cartan matrix Ω of G
-

G	A_r	B_r	$(C_r)_0$	$(C_r)_\pi$	D_r	E_6	E_7	E_8	F_4	G_2
$\#\vec{\lambda}_G$	$r+1$	2	2	1	4	3	2	1	1	1

$$\lambda_F = \pm \frac{1}{2}$$

- For a given gauge group, there are many unity and vanishing blow up formulas.
- They are highly non-trivial consistent conditions on Gopakumar-Vafa invariants. KL,K.Sun to appear
- For some cases, the 1-loop corrected prepotential $\mathscr{F}_{\text{prepot}}$ determines all instanton corrections.

Elliptic Genus for Partition Function on $T^2 \times \mathbb{R}^4_{\epsilon_{1,2}}$

Two approaches to the partition function

Counting BPS states on $R^4_{\epsilon_1,\epsilon_2} \times T^2$

5d | **6d**

$$Z_{\text{BPS}} = Z_{\text{pert}}\left(1 + \sum_k Z_k(\mathbf{v})q^k\right) = Z_0\left(1 + \sum_{\mathbf{n}} \mathbb{E}_{\mathbf{n}}(q)e^{-\mathbf{n}\cdot\mathbf{v}}\right)$$

Instanton Counting		**Elliptic Genus**
No known ADHM for exceptional Lie Groups	Difficulties	No known brane diagrams except A, D types
Blow Ups	**Solutions**	**Modular Bootstrap**

Free part Z_0

Free (2,0) tensor modes

$$Z_0(G) = \text{PE}\left[\frac{\sinh\frac{m\pm\epsilon_-}{2}}{\sinh\frac{\epsilon_{1,2}}{2}}\chi(q)_G\right]$$

$$\chi_{U(1),A_r,D_r,E_r} = \frac{rq}{1-q},\ \chi_{B_r} = \frac{(r-1)q}{1-q} + \frac{q^2}{1-q^2},\ \chi_{(C_r)_0} = \frac{(r-1)q^2}{1-q^2} + \frac{q}{1-q},$$

$$\chi_{(C_r)_\pi} = \frac{(r-1)q^2}{1-q^2} + \frac{q^4}{1-q^4} + \frac{q^2}{1-q^2},\ \chi_{G_2} = \frac{q}{1-q} + \frac{q^3}{1-q^3},\ \chi_{F_4} = \frac{2q}{1-q} + \frac{2q^2}{1-q^2}$$

Elliptic Genus $\mathbb{E}_{\mathbf{n}}$

- Jacobi form $\phi_{k,m}(\tau, z)$

k: weight; m: index

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}), \qquad \phi_{k,m}\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = (c\tau+d)^k e^{2\pi i c \frac{(m\, z^2)}{c\tau+d}}\, \phi_{k,m}(\tau, z)$$

$$\phi_{k,m}(\tau, z+\lambda\tau+\mu) = e^{-2\pi i m\left((\lambda,\lambda)\tau+2(\lambda,z)\right)}\, \phi_{k,m}(\tau, z) \qquad \lambda,\mu \in \mathbb{Z}$$

- $\mathbb{E}_{\mathbf{n}} = \mathbb{E}_{\mathbf{n}}(\tau, \epsilon_1, \epsilon_2, m)$: Jacobi form of weight 0

Index = Anomaly Poly. $\quad \mathfrak{i}_{\mathbf{n}} = \frac{\epsilon_1\epsilon_2}{2}\mathbf{n}^T \cdot \Omega \cdot \mathbf{n} + \left(m^2 - \frac{(\epsilon_1+\epsilon_2)^2}{4}\right)\sum_{a=1}^{r} n_a$

Anomaly Inflow
[Shimizu and Tachikawa, 1608.05894]
Holomorphic Anomaly
[Huang, Katz and Klemm, 1501.04891]

Bootstrap for Elliptic Genus of ADE case

[Gu et al, 1701.00764] [Del Zotto et al, 1712.07017]
[Del Zotto and Lockhart, 1609.00310, 1804.09694] [Kim,KL, Park, 1801.01631]
[Lee, Lerche and Weigand 1808.05958] [Duan, Gu and Kashani-Poor, 1810.01280]
[Duan, Jaramillo Duque and Kashani-Poor, 2012.10427]
[Duan,Nahmgoong, 2009.03626]

Meromorphic Form

$$\mathbb{E}_{\mathbf{n}} = \frac{\mathcal{N}_{\mathbf{n}}(\tau, m, \epsilon_{1,2})}{\mathcal{D}_{\mathbf{n}}(\tau, \epsilon_{1,2})}, \quad \mathcal{N}_{\mathbf{n}}(\tau, m_f, \epsilon_{1,2}) \in \mathbb{C}[E_4, E_6][A_{-2,1}(\tau; m, \epsilon_\pm), B_{0,1}(\tau; m, \epsilon_\pm)]$$

The poles can be inferred from Nekrasov partition function in 5d

Generators of $SU(2)$ weak Jacobi forms

Index of generators is always positive: finite dimensional problem!

$$A(\tau,z) = \varphi_{-2,1}(\tau,z) = \frac{\theta_1(\tau,z)^2}{\eta(\tau)^6} \qquad B(\tau,z) = \varphi_{0,1}(\tau,z) = 4\left(\frac{\theta_2(\tau,z)^2}{\theta_2(\tau,0)^2} + \frac{\theta_3(\tau,z)^2}{\theta_3(\tau,0)^2} + \frac{\theta_4(\tau,z)^2}{\theta_4(\tau,0)^2}\right)$$

Twisting and identification of self-dual strings

- Untwisted: 5d MSYM with ADE gauge group

- $\mathbb{Z}_{n_G}$ Twist: 5d MSYM with BCC'FG gauge group

$$
\begin{array}{lll}
\mathbb{Z}_2: \; B_r: \; \mathbf{n} = (n_1, n_2, ..., n_r) & \leftarrow & A_{2r-1}: \; \mathbf{n} = (n_1, n_2, ..., n_r, n_{r-1}, ..., n_1) \\
\mathbb{Z}_2: \; (C_r)_0: \; \mathbf{n} = (n_1, n_2, ..., n_r) & \leftarrow & D_{r+1}: \; \mathbf{n} = (n_1, n_2, ..., n_{r-1}, n_r, n_r) \\
\mathbb{Z}_3: \; G_2: \; \mathbf{n} = (n_1, n_2) & \leftarrow & D_4: \; \mathbf{n} = (n_2, n_1, n_2, n_2) \\
\mathbb{Z}_2: \; F_4: \; \mathbf{n} = (n_1, n_2, n_3, n_4) & \leftarrow & E_6: \; \mathbf{n} = (n_1, n_2, n_3, n_2, n_1, n_4) \\
\mathbb{Z}_4: \; (C_r)_\pi: \; \mathbf{n} = (n_1, n_2, ..., n_r) & \leftarrow & A_{2r}: \; \mathbf{n} = (n_1, ..., n_r, n_r, ..., n_1)
\end{array}
$$

$\downarrow$ Folding the tensor multiplets, not the gauge algebra

$\Pi_4(C_r) = \mathbb{Z}_2$

Genus one fibration with n_G-section

$$
\begin{array}{ccc}
\tilde{T}^2 & \longrightarrow & \tilde{X} \\
 & & \downarrow \pi \\
 & & B
\end{array}
$$

"Folding" ADE singularity

[Bhardwaj et al, 1909.11666]

- Twist along S^1: $\mathbb{E}^{tw}_{\mathbf{n}} = Z^{ref}_{\mathbf{n}}$ top. string partition function on $\tilde{X}$

Partition Function on $R^4_{\epsilon_{1,2}} \times S^1_T \times S^1_M$

After folding, fundamental strings with possible KK momenta

Twist	5d $G^\vee$	Long	Short	Special
$A_{2r}^{(2)}$	$(C_r)_\pi$	$[\alpha_r]_{2k,4k}$	$[\alpha_a]_{2k}$	$[\frac{1}{2}\alpha_r]_{4k\pm 1}$
$A_{2r-1}^{(2)}$	B_r	$[\alpha_a]_k$	$[\alpha_r]_{2k}$	—
$D_{r+1}^{(2)}$	$(C_r)_0$	$[\alpha_r]_k$	$[\alpha_a]_{2k}$	—
$E_6^{(2)}$	F_4	$[\alpha_1]_k, [\alpha_2]_k$	$[\alpha_3]_{2k}, [\alpha_4]_{2k}$	—
$D_4^{(3)}$	G_2	$[\alpha_1]_k$	$[\alpha_2]_{3k}$	—

Twisted case

[Duan, KL, Nahmgoong, Wang, 2103.06044]

- Fold the index of untwisted $\mathbb{E}_{\mathbf{n}}$ $\quad \mathfrak{i}_{\mathbf{n}}(z) \text{ of } G^\vee = \frac{1}{n_G}\left(\mathfrak{i}_{\mathbf{n}}(z) \text{ of } G\right) = \frac{\epsilon_1\epsilon_2}{2}\mathbf{n}^T \cdot (\Omega^{\mathsf{S}}) \cdot \mathbf{n} + (m^2 - \epsilon_+^2)\sum_{a=1}^{r} D_{aa} n_a$ Fractional!

$\Omega^{\mathsf{S}} \to 4\frac{\langle\alpha_i,\alpha_j\rangle}{\langle\alpha_i,\alpha_i\rangle\langle\alpha_j,\alpha_j\rangle}$ Symmetrized Cartan matrix of $G^\vee$

$D_{aa} \to \frac{2}{\langle\alpha_a,\alpha_a\rangle}$

Meromorphic Form

- $\mathbb{E}_{\mathbf{n}}^{tw} = \frac{\mathcal{N}_{\mathbf{n}}(\tau, m_f, \epsilon_{1,2})}{\mathcal{D}_{\mathbf{n}}(\tau, \epsilon_{1,2})} = \sum_k f^{(k)}(\tau, z) \cdot \hat{f}^{(k)}(n_G\tau, z)$

- The congruence group plays a role here.
  - $\Gamma_0(n_G) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) : c \equiv 0 \pmod{n_G} \right\}$

$\Omega^S = \Omega D$

[Duan, KL, Nahmgoong, Wang, 2103.06044]

B_n	$(C_r)_0$	F_4	G_2
$\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & 2 & -1 \\ 0 & \cdots & \cdots & 0 & -1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & -\frac{1}{2} & 0 & \cdots & 0 & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} & \cdots & 0 & 0 \\ 0 & -\frac{1}{2} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & 1 & -1 \\ 0 & \cdots & \cdots & 0 & -1 & 2 \end{pmatrix}$	$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 1 & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 1 \end{pmatrix}$	$\begin{pmatrix} \frac{2}{3} & -1 \\ -1 & 2 \end{pmatrix}$

$(C_r)_\pi$

$$\begin{pmatrix} 1 & -\frac{1}{2} & 0 & \cdots & 0 & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} & \cdots & 0 & 0 \\ 0 & -\frac{1}{2} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & 1 & -\frac{1}{2} \\ 0 & \cdots & \cdots & 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Modular Bootstrap Elliptic Genus $\mathbb{E}_{\mathbf{n}}$

- Vanishing Bound: $N^{\beta}_{j_-,j_+}$ always vanishes whenever either j_- or $j_+ \gg 0.$

[Pandharipande and Thomas, 0707.2348][Choi, Katz and Klemm, 1210.4403],

- Expand Free Energy in GV invariants: $\mathcal{F}_{GV} = \sum_{\mathbf{v}} F_{\mathbf{n}} e^{-\mathbf{n}\cdot\mathbf{v}}$

    - (We know a few of them perturbative corrections from 5d)

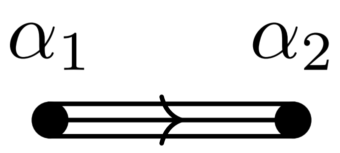
- Expand PE exponent: $\mathcal{Z}_{GV} = \text{PE}(\mathcal{F}_{GV}) = \exp(\mathcal{F}) = Z_{PE}^{\text{1-loop}}(1 + \sum_{\mathbf{n}} \mathbb{E}_{\mathbf{n}})$

- Solve recursively $\mathbb{E}_{\mathbf{n}}$: unique upto few exceptions for small $\mathbf{n}$

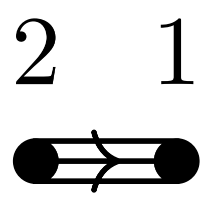
Elliptic Genus for 5d G_2

[ZD, Lee, Nahmgoong and Wang, 2103.06044]

5d G_2 MSYM: $\mathbb{Z}_3$ twisted elliptic genus



$\mathbf{n} =$



$$\mathscr{D}_{\mathbf{n}}(\tau,\epsilon_{1,2}) = \frac{\theta_1(3\tau,\epsilon_{1,2})}{\eta(3\tau)^3}\frac{\theta_1(\tau,\epsilon_{1,2})}{\eta(\tau)^3}\frac{\theta_1(\tau,2\epsilon_{1,2})}{\eta(\tau)^3}$$

$$\mathfrak{j}_{\mathbf{n}} = -\frac{7}{3}\epsilon_-^2 + \frac{14}{3}m_f^2$$

$$\hat{f}(\tau) := f(3\tau)$$

$$\mathcal{N}_{\mathbf{n}}(\tau, m, \epsilon_{1,2}) =$$

$$\frac{1}{2^{17}3^9}(\widehat{A}_m\widehat{B}_- - \widehat{A}_-\widehat{B}_m)(\widehat{A}_m\widehat{B}_+ - \widehat{A}_+\widehat{B}_m)\Big(-27B_+A_m\widehat{E}_4^3\widehat{A}_-^6 + 27A_+B_m\widehat{E}_4^3\,\widehat{A}_-^6 + 32B_+A_m\widehat{E}_6^2\widehat{A}_-^6 - 32A_+B_m\widehat{E}_6^2\,\widehat{A}_-^6$$
$$+24\widehat{B}_-B_+A_m\widehat{E}_4\widehat{E}_6\widehat{A}_-^5 - 24\,\widehat{B}_-A_+B_m\widehat{E}_4\widehat{E}_6\widehat{A}_-^5 + 45\widehat{B}_-^2B_+\,A_m\widehat{E}_4^2\widehat{A}_-^4 - 45\widehat{B}_-^2A_+B_m\widehat{E}_4^2\,\widehat{A}_-^4 + 54B_-\widehat{A}_+^3A_m\widehat{E}_4^3\widehat{A}_-^3 - 54A_-\,\widehat{A}_+^3B_m\widehat{E}_4^3\widehat{A}_-^3$$
$$-18B_-\widehat{A}_+\widehat{B}_+^2\,A_m\widehat{E}_4^2\widehat{A}_-^3 + 18A_-\widehat{A}_+\widehat{B}_+^2B_m\,\widehat{E}_4^2\widehat{A}_-^3 - 64B_-\widehat{A}_+^3A_m\widehat{E}_6^2\,\widehat{A}_-^3 + 64A_-\widehat{A}_+^3B_m\widehat{E}_6^2\widehat{A}_-^3 - 4B_-\,\widehat{B}_+^3A_m\widehat{E}_6\widehat{A}_-^3 + 40\widehat{B}_-^3B_+A_m\,\widehat{E}_6\widehat{A}_-^3$$
$$+4A_-\widehat{B}_+^3B_m\widehat{E}_6\widehat{A}_-^3 - 40\,\widehat{B}_-^3A_+B_m\widehat{E}_6\widehat{A}_-^3 - 24B_-\widehat{A}_+^2\,\widehat{B}_+A_m\widehat{E}_4\widehat{E}_6\widehat{A}_-^3 + 24A_-\widehat{A}_+^2\,\widehat{B}_+B_m\widehat{E}_4\widehat{E}_6\widehat{A}_-^3 - 54B_-\widehat{B}_-\,\widehat{A}_+^2\widehat{B}_+A_m\widehat{E}_4^2\widehat{A}_-^2 + 54A_-\widehat{B}_-\,\widehat{A}_+^2\widehat{B}_+B_m\widehat{E}_4^2\widehat{A}_-^2 - 6B_-\widehat{B}_-\,\widehat{B}_+^3A_m\widehat{E}_4\widehat{A}_-^2$$
$$+15\widehat{B}_-^4B_+A_m\,\widehat{E}_4\widehat{A}_-^2 + 6A_-\widehat{B}_-\widehat{B}_+^3B_m\widehat{E}_4\,\widehat{A}_-^2 - 15\widehat{B}_-^4A_+B_m\widehat{E}_4\widehat{A}_-^2 - 36B_-\,\widehat{B}_-\widehat{A}_+\widehat{B}_+^2A_m\widehat{E}_6\widehat{A}_-^2 + 36A_-\,\widehat{B}_-\widehat{A}_+\widehat{B}_+^2B_m\widehat{E}_6\widehat{A}_-^2 - 24B_-\,\widehat{B}_-\widehat{A}_+^3A_m\widehat{E}_4\widehat{E}_6\widehat{A}_-^2$$
$$+24A_-\,\widehat{B}_-\widehat{A}_+^3B_m\widehat{E}_4\widehat{E}_6\widehat{A}_-^2 - 18B_-\,\widehat{B}_-^2\widehat{A}_+^3A_m\widehat{E}_4^2\widehat{A}_- + 18A_-\widehat{B}_-^2\,\widehat{A}_+^3B_m\widehat{E}_4^2\widehat{A}_- - 18B_-\widehat{B}_-^2\widehat{A}_+\,\widehat{B}_+^2A_m\widehat{E}_4\widehat{A}_- + 18A_-\widehat{B}_-^2\widehat{A}_+\,\widehat{B}_+^2B_m\widehat{E}_4\widehat{A}_-$$
$$-36B_-\widehat{B}_-^2\widehat{A}_+^2\,\widehat{B}_+A_m\widehat{E}_6\widehat{A}_- + 36A_-\widehat{B}_-^2\widehat{A}_+^2\,\widehat{B}_+B_m\widehat{E}_6\widehat{A}_- + 2B_-\widehat{B}_-^3\widehat{B}_+^3\,A_m - \widehat{B}_-^6B_+A_m - 2A_-\widehat{B}_-^3\widehat{B}_+^3B_m + \widehat{B}_-^6\,A_+B_m - 6B_-\widehat{B}_-^3\widehat{A}_+^2\widehat{B}_+A_m\widehat{E}_4 + 6A_-\,\widehat{B}_-^3\widehat{A}_+^2\widehat{B}_+B_m\widehat{E}_4 - 4B_-\widehat{B}_-^3\,\widehat{A}_+^3A_m\widehat{E}_6$$
$$+4A_-\widehat{B}_-^3\widehat{A}_+^3B_m\widehat{E}_6\Big)$$

ring structure for numerator

Gauge group	Base degree	Index	Weight	Unknowns
G_2	0 1	$\epsilon_+^2 + m^2$	-2	2
G_2	1 0	$\frac{1}{3}\epsilon_+^2 + \frac{1}{3}m^2$	-2	2
G_2	1 1	$\frac{1}{3}\epsilon_+^2 + \epsilon_-^2 + \frac{4}{3}m^2$	-4	4
G_2	1 2	$\frac{16}{3}\epsilon_+^2 + 3\epsilon_-^2 + \frac{7}{3}m^2$	-6	132
G_2	2 1	$\frac{4}{3}\epsilon_+^2 + \frac{7}{3}\epsilon_-^2 + \frac{5}{3}m^2$	-6	226
B_3	1 1 1	$\frac{1}{2}\epsilon_+^2 + 2\epsilon_-^2 + \frac{5}{2}m^2$	-6	8
B_3	2 1 1	$\frac{11}{2}\epsilon_+^2 + 4\epsilon_-^2 + \frac{7}{2}m^2$	-8	220
B_3	1 2 1	$\frac{9}{2}\epsilon_+^2 + 5\epsilon_-^2 + \frac{7}{2}m^2$	-8	220
$(C_3)_0$	1 1 1	$\frac{9}{2}\epsilon_+^2 + 5\epsilon_-^2 + \frac{7}{2}m^2$	-6	10
$(C_3)_0$	1 1 2	$\frac{11}{2}\epsilon_+^2 + \frac{7}{2}\epsilon_-^2 + 3m^2$	-8	330
F_4	1 1 1 1	$\frac{1}{2}\epsilon_+^2 + \frac{5}{2}\epsilon_-^2 + 3m^2$	-8	20
F_4	2 1 1 1	$\frac{11}{2}\epsilon_+^2 + \frac{9}{2}\epsilon_-^2 + 4m^2$	-10	550
F_4	1 2 1 1	$\frac{9}{2}\epsilon_+^2 + \frac{11}{2}\epsilon_-^2 + 4m^2$	-10	550

Numerator ring structure for $(C_r)_\pi$

$$\mathbb{E}_{\mathfrak{n}} = \frac{\mathcal{N}_{\mathfrak{n}}}{\mathcal{D}_{\mathfrak{n}}}, \quad \mathcal{D}_{\mathfrak{n}} = \prod_{a=1}^{r} \prod_{k=1}^{n_a} \frac{\theta_1(2\tau, k\epsilon_1)}{\eta^3(2\tau)} \cdot \frac{\theta_1(2\tau, k\epsilon_2)}{\eta^3(2\tau)} \qquad w_{\mathfrak{n}} = 0, \quad i_{\mathfrak{n}} = \frac{\epsilon_1\epsilon_2}{2}\mathfrak{n}^T(\Omega^s)\mathfrak{n} + \frac{1}{2}(m^2 - \epsilon_+^2)\sum_{a=1}^{r}\mathfrak{n}_a$$

- 1 $$\mathbb{E}_1(\tau, \epsilon_1, \epsilon_2, m) = \frac{(E_2^{(2)} - E_2^{(4)})(\widehat{A}_m\widehat{B}_+ - \widehat{A}_+\widehat{B}_m)(\widehat{A}_m\widehat{B}_- - \widehat{A}_-\widehat{B}_m)}{2^8\, 3^2\, \varphi_{-2,1}(2\tau, \epsilon_1)\varphi_{-2,1}(2\tau, \epsilon_2)}$$
- 2 $$\mathbb{E}_2^{(C_1)_\pi}(\tau, \epsilon_1, \epsilon_2, m) = \frac{1}{2}\left(\mathbb{E}_1^{(C_1)_0}(\tau, \epsilon_1, \epsilon_2, m) + \mathbb{E}_1^{(C_1)_0}(\tau + 1/2, \epsilon_1, \epsilon_2, m)\right) + \frac{(\widehat{A}_m\widehat{B}_+ - \widehat{A}_+\widehat{B}_m)^2(\widehat{A}_m\widehat{B}_- - \widehat{A}_-\widehat{B}_m)^2}{\prod_{k=1}^{2}\varphi_{-2,1}(2\tau, k\epsilon_1)\varphi_{-2,1}(2\tau, k\epsilon_2)}\mathcal{I}(\tau, \epsilon_1, \epsilon_2)$$

$$\tfrac{1}{2^{63}3^{16}}\Big(E_{2,2} - E_{2,4}\Big)^2\Big(8E_{2,2}^3\widehat{A}_-\widehat{A}_+^2\widehat{B}_- + 8E_{2,2}^3\widehat{A}_-^2\widehat{A}_+\widehat{B}_+ + 16E_{2,2}^2\widehat{A}_+^2\widehat{B}_-^2 + 16E_{2,2}^2\widehat{A}_-^2\widehat{B}_+^2 + 64E_{2,2}^2\widehat{A}_-\widehat{A}_+\widehat{B}_-\widehat{B}_+ + 128E_{2,2}\widehat{A}_-\widehat{B}_-\widehat{B}_+^2 + 128E_{2,2}\widehat{A}_+\widehat{B}_-^2\widehat{B}_+ + E_{2,2}^4\widehat{A}_-^2\widehat{A}_+^2 + 256\widehat{B}_-^2\widehat{B}_+^2$$
$$+1224E_{2,4}E_{2,2}^2\widehat{A}_-\widehat{A}_+^2\widehat{B}_- + 1224E_{2,4}E_{2,2}^2\widehat{A}_-^2\widehat{A}_+\widehat{B}_+ + 96E_{2,4}E_{2,2}\widehat{A}_+^2\widehat{B}_-^2 + 96E_{2,4}E_{2,2}\widehat{A}_-^2\widehat{B}_+^2 - 1920E_{2,4}E_{2,2}\widehat{A}_-\widehat{A}_+\widehat{B}_-\widehat{B}_+ + 144E_{2,4}^2\widehat{A}_+^2\widehat{B}_-^2 + 384E_{2,4}\widehat{A}_-\widehat{B}_-\widehat{B}_+^2 + 384E_{2,4}\widehat{A}_+\widehat{B}_-^2\widehat{B}_+ - 564E_{2,4}E_{2,2}^3\widehat{A}_-^2\widehat{A}_+^2$$
$$+3672E_{2,4}^3\widehat{A}_-\widehat{A}_+^2\widehat{B}_- + 3672E_{2,4}^3\widehat{A}_-^2\widehat{A}_+\widehat{B}_+ + 144E_{2,4}^2\widehat{A}_-^2\widehat{B}_+^2 + 2880E_{2,4}^2\widehat{A}_-\widehat{A}_+\widehat{B}_-\widehat{B}_+ - 4392E_{2,2}E_{2,4}^2\widehat{A}_-\widehat{A}_+^2\widehat{B}_- - 4392E_{2,2}E_{2,4}^2\widehat{A}_-^2\widehat{A}_+\widehat{B}_+ + 5265E_{2,4}^4\widehat{A}_-^2\widehat{A}_+^2 - 8532E_{2,2}E_{2,4}^3\widehat{A}_-^2\widehat{A}_+^2 + 4086E_{2,2}^2E_{2,4}^2\widehat{A}_-^2\widehat{A}_+^2\Big)$$
$$\Big(152E_{2,2}^3\widehat{A}_-\widehat{A}_+^2\widehat{B}_- + 152E_{2,2}^3\widehat{A}_-^2\widehat{A}_+\widehat{B}_+ + 400E_{2,2}^2\widehat{A}_+^2\widehat{B}_-^2 + 400E_{2,2}^2\widehat{A}_-^2\widehat{B}_+^2 - 704E_{2,2}^2\widehat{A}_-\widehat{A}_+\widehat{B}_-\widehat{B}_+ - 640E_{2,2}\widehat{A}_-\widehat{B}_-\widehat{B}_+^2 - 640E_{2,2}\widehat{A}_+\widehat{B}_-^2\widehat{B}_+ + 49E_{2,2}^4\widehat{A}_-^2\widehat{A}_+^2 + 256\widehat{B}_-^2\widehat{B}_+^2 - 1512E_{2,4}E_{2,2}^2\widehat{A}_-\widehat{A}_+^2\widehat{B}_-$$
$$-1512E_{2,4}E_{2,2}^2\widehat{A}_-^2\widehat{A}_+\widehat{B}_+ - 1440E_{2,4}E_{2,2}\widehat{A}_+^2\widehat{B}_-^2 - 1440E_{2,4}E_{2,2}\widehat{A}_-^2\widehat{B}_+^2 + 1152E_{2,4}E_{2,2}\widehat{A}_-\widehat{A}_+\widehat{B}_-\widehat{B}_+ + 1296E_{2,4}^2\widehat{A}_+^2\widehat{B}_-^2 + 1152E_{2,4}\widehat{A}_-\widehat{B}_-\widehat{B}_+^2 + 1152E_{2,4}\widehat{A}_+\widehat{B}_-^2\widehat{B}_+ + 684E_{2,4}E_{2,2}^3\widehat{A}_-^2\widehat{A}_+^2$$
$$-1080E_{2,4}^3\widehat{A}_-\widehat{A}_+^2\widehat{B}_- - 1080E_{2,4}^3\widehat{A}_-^2\widehat{A}_+\widehat{B}_+ + 1296E_{2,4}^2\widehat{A}_-^2\widehat{B}_+^2 + 576E_{2,4}^2\widehat{A}_-\widehat{A}_+\widehat{B}_-\widehat{B}_+ + 2952E_{2,2}E_{2,4}^2\widehat{A}_-\widehat{A}_+^2\widehat{B}_- + 2952E_{2,2}E_{2,4}^2\widehat{A}_-^2\widehat{A}_+\widehat{B}_+ - 3807E_{2,4}^4\widehat{A}_-^2\widehat{A}_+^2 + 7884E_{2,2}E_{2,4}^3\widehat{A}_-^2\widehat{A}_+^2 - 4554E_{2,2}^2E_{2,4}^2\widehat{A}_-^2\widehat{A}_+^2\Big)$$
$$\Big(16E_{2,2}^3\widehat{A}_+^3\widehat{B}_+ - 1872E_{2,2}^2E_{2,4}\widehat{A}_+^3\widehat{B}_+ + 96E_{2,2}^2\widehat{A}_+^2\widehat{B}_+^2 + 2880E_{2,2}E_{2,4}\widehat{A}_+^2\widehat{B}_+^2 + 256E_{2,2}\widehat{A}_+\widehat{B}_+^3 - 768E_{2,4}\widehat{A}_+\widehat{B}_+^3 + E_{2,2}^4\widehat{A}_+^4 + 636E_{2,2}^3E_{2,4}\widehat{A}_+^4 + 256\widehat{B}_+^4 - 10800E_{2,4}^3\widehat{A}_+^3\widehat{B}_+ + 9072E_{2,2}E_{2,4}^2\widehat{A}_+^3\widehat{B}_+$$
$$-6048E_{2,4}^2\widehat{A}_+^2\widehat{B}_+^2 - 6399E_{2,4}^4\widehat{A}_+^4 + 8316E_{2,2}E_{2,4}^3\widehat{A}_+^4 - 3834E_{2,2}^2E_{2,4}^2\widehat{A}_+^4\Big)$$

Cardy Limit

Large charge limit

KL, J. Nahmgoong, 2006.10294

- The partition function $Z(\epsilon_1, \epsilon_2, \tau)$ of 6d ADE theories on $\mathbb{R}^4 \times T^2$
- large angular momenta = $|\epsilon_{1,2}| \to 0$: vol($\mathbb{R}^4$) $\sim \dfrac{1}{\epsilon_1 \epsilon_2}$
- large Kaluza-Klein momenta= $|\tau| \to 0$: vol(R_{KK}) $\sim \dfrac{1}{\tau}$
- The single particle free energy $Z =$ PE($\mathcal{F}_s$): $\mathcal{F}_s \sim \mathcal{O}\left(\dfrac{1}{\epsilon_1 \epsilon_2 \tau}\right)$
- Complex chemical potentials: Re(ϵ_1) > 0, Re(ϵ_2) < 0, Re($2\pi i \tau$) < 0
    - $J_1 + Q_R \ll 0, \;\; J_2 + Q_R \gg 0, \;\; P \gg 0$

Modular Transformation: $\tau \rightarrow -1/\tau$

- Elliptic genus generic form: $\mathbb{E}_n(\tau,z) = \dfrac{1}{D_{\mathbf{n}}(\tau,z)\cdot \hat{D}_{\mathbf{n}}(n_G\tau,z)} \times \sum_{a=1}^{|S|} N_{\mathbf{n}}^{(a)}(\tau,z)\cdot \hat{N}_{\mathbf{n}}(n_G\tau,z)$
- The modular group is $\Gamma_0(n_G) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}), c \equiv 0 \text{ mod } n_G \right\}$
- Ellipitic genus for $\mathbf{n}$ strings in G (except $(C_r)_\pi$)

$$\mathbb{E}_{\mathbf{n}}(-\frac{1}{\tau},\frac{z}{\tau}) = t^{w+\hat{w}} \exp\left[-\frac{\pi i}{\tau}\left(\mathfrak{j}(z)+\frac{\hat{\mathfrak{j}}(z)}{N}\right)\right]\cdot \mathbb{E}_{\mathbf{n}}(\tau,z) = \exp\left[-\frac{\pi i}{\tau}\mathfrak{i}_{\mathbf{n}}(z)\right]\cdot \mathbb{E}_{\mathbf{n}}(\tau,z)$$

- Invert it to get $\mathbb{E}_{\mathbf{n}}(\tau,z)$ in small $\tau \rightarrow i0^+$ limit or $\exp(-2\pi i/\tau) \rightarrow 0$ limit.

Cardy Limit:

- In small $\tau \to i0^+$ limit,
  - $\mathbb{E}_{\mathbf{n}}(\tau, z) \simeq \exp\Big[-\frac{1}{2\pi\tau}\mathfrak{i}(\tau, z)\Big] \times \Big(1 + \exp(-a/\tau)\cdots\Big)$
  - $\mathbb{E}_{\mathbf{n}}(\tau, z) \simeq \exp\Big[-\frac{1}{2\pi i\tau}\Big(\frac{\epsilon_1\epsilon_2}{2}\mathbf{n}^T(\Omega D)\mathbf{n} + m(m - 2\pi i)\mathbf{D}\cdot\mathbf{n}\Big)\Big] \cdot \Big(1 + \cdots\Big)$
- $\hat{Q}^D_{1,2} = e^{-\frac{\epsilon_{1,2}}{n_G\tau}}$, $\hat{Q}^D_\tau = e^{-\frac{2\pi i}{n_G\tau}}$, $\hat{Q}^D_m = e^{-\frac{m}{n_G\tau}}$: $\hat{Q}^D_\tau \ll \hat{Q}^D_m \ll 1,\ \hat{Q}^D_{1,2} \sim 1$
- The saddle point for $Z \sim \sum_{\mathbf{n}} e^{=\mathbf{v}\cdot\mathbf{n}} \exp\Big[-\frac{1}{2\pi i\tau}\Big(\frac{\epsilon_1\epsilon_2}{2}\mathbf{n}^T(\Omega D)\mathbf{n} + m(m - 2\pi i)\mathbf{D}\cdot\mathbf{n}\Big)\Big]$

Saddle point:

- With $\mathfrak{r} = -\epsilon_1\epsilon_2 D\mathfrak{n}$ for small $\epsilon_{1,2}$ limit , the sum over $\mathfrak{n}$ becomes the integration over $\mathfrak{r}$
  - The saddle point $\mathfrak{r} = m(m-2\pi i)(\Omega^{-1}D)I$ where r-dim vector $I = (1,1,\ldots,1)^T$.
  - $\sum_{a,b=1}^{r} (D^{-1}\Omega^{-1}D)_{ab}\alpha_a = \frac{1}{2}\sum_{\boldsymbol{\alpha}\in\Delta^+}\boldsymbol{\alpha} = \boldsymbol{\rho}$ with Weyl vector $\boldsymbol{\rho}$
  - $\boldsymbol{\rho}\cdot\boldsymbol{\rho} = \frac{h_G^\vee \cdot d_G}{12} = I^T \cdot \Omega^{-1} \cdot I,$
- The saddle point string number $\langle \mathbf{n} \rangle = \sum_a \langle n_a \rangle \alpha_a = \frac{m(2\pi i - m)}{\epsilon_1\epsilon_2}\boldsymbol{\rho}$
- The log of the partition function
  - $\log Z \simeq -\frac{h_G d_G}{24}\frac{m^2(2\pi i - m)^2}{(-2\pi i\tau)\epsilon_1\epsilon_2}$

$$d_G,\ h_G^\vee,\ \boldsymbol{\rho}\cdot\boldsymbol{\rho} = \frac{h_G^\vee d_G}{12}$$

	d_G	$h_G^\vee$	$\lvert\vec{\rho}\rvert^2$
A_r	$(r+1)^2-1$	$r+1$	$\frac{1}{12}r^3+\frac{1}{4}r^2+\frac{1}{6}r$
B_r	$2r^2+r$	$2r-1$	$\frac{1}{3}r^3-\frac{1}{12}r$
C_r	$2r^2+r$	$r+1$	$\frac{1}{6}r^3+\frac{1}{4}r^2+\frac{1}{12}r$
D_r	$2r^2-r$	$2r-2$	$\frac{1}{3}r^3-\frac{1}{2}r^2+\frac{1}{6}r$
G_2	14	4	$\frac{14}{3}$
F_4	52	9	39
E_6	78	12	78
E_7	133	18	$\frac{399}{2}$
E_8	248	30	620

Large Charge Limit

- $J_- = J_1 - J_2 > 0,\ |J_-| \gg |J_+ + Q_R|, |Q_m|\ ,\ \ P > 0$
- $S \simeq \log Z + \mathbf{n} \cdot \mathbf{v} + \epsilon_1(J_1 + Q_R) + \epsilon_2(J_2 + Q_R) + \beta P + 2mQ_m,\ \ \beta = -2\pi i\tau$
- $S \simeq 2\pi\sqrt{\sqrt{-\frac{2}{3}h_G^\vee d_G P(J_1 + Q_R)(J_2 + Q_R)} - Q_m^2} + 2\pi i Q_m$
- Black string entropy. $\mathrm{Re}(S) = 2\pi\sqrt{\sqrt{\frac{2}{3}h_G d_G P(J_-^2 - (J_+ + Q_R)^2)} - Q_m^2}$

Invariance of D.O.F under twisting

- Cardy Limit= $|\tau| \ll 1$, the large KK momentum or short distant physics become dominant.
- Thus the compactification, that is, twisting is not relevant.
- The degrees of freedom should be invariant.
- $\frac{h_{G_0} d_{G_0}}{n_{G_0}} = h_{G^\vee} d_{G^\vee}$ (true!)
- $R_{G_0} = n_{G_0} R_{G^\vee}$ for integer KK momenta on $\frac{1}{R_{G^\vee}}$:
- $h_{G_0} d_{G_0} P_{G_0} = h_{G^\vee} d_{G^\vee} P_{G^\vee}$
- For E6/Z2=F4, 78*12/2=52*9

	d_G	$h_G^\vee$	$\|\vec{\rho}\|^2$
A_r	$(r+1)^2-1$	$r+1$	$\frac{1}{12}r^3+\frac{1}{4}r^2+\frac{1}{6}r$
B_r	$2r^2+r$	$2r-1$	$\frac{1}{3}r^3-\frac{1}{12}r$
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Conclusion

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- We did not covered the compactification of 6d (1,0) theories.
  - Primary interest has been to get more 5d SCFTs from 6d SCFTs
  - Their various properties, such as partition functions, 5d counter parts, integrable models, BPS quivers, discrete symmetries, defects partition function, AdS-CFT correspondence, and Cardy limit are all very interesting.
- Future directions:
  - partition function with codimension 2 and 4 defect

    Chen,Haghighat,Kim,Sperling('20):
    Chen Haghghat,Kim,Sperling,Wang('21),
    Chen,Haghghat,Kim,KL,Sperling,Wang('21)
  - NS limits and quantization of the SW curve
  - Twisting of (2,0) and (1,1) little string theories
  - lower dimensional field theories from 6d and 5d SCFTs