Weak Cosmic Censorship for Higher Derivative Gravity Theories

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based on 1902.00949, 2006.08663 & on-going
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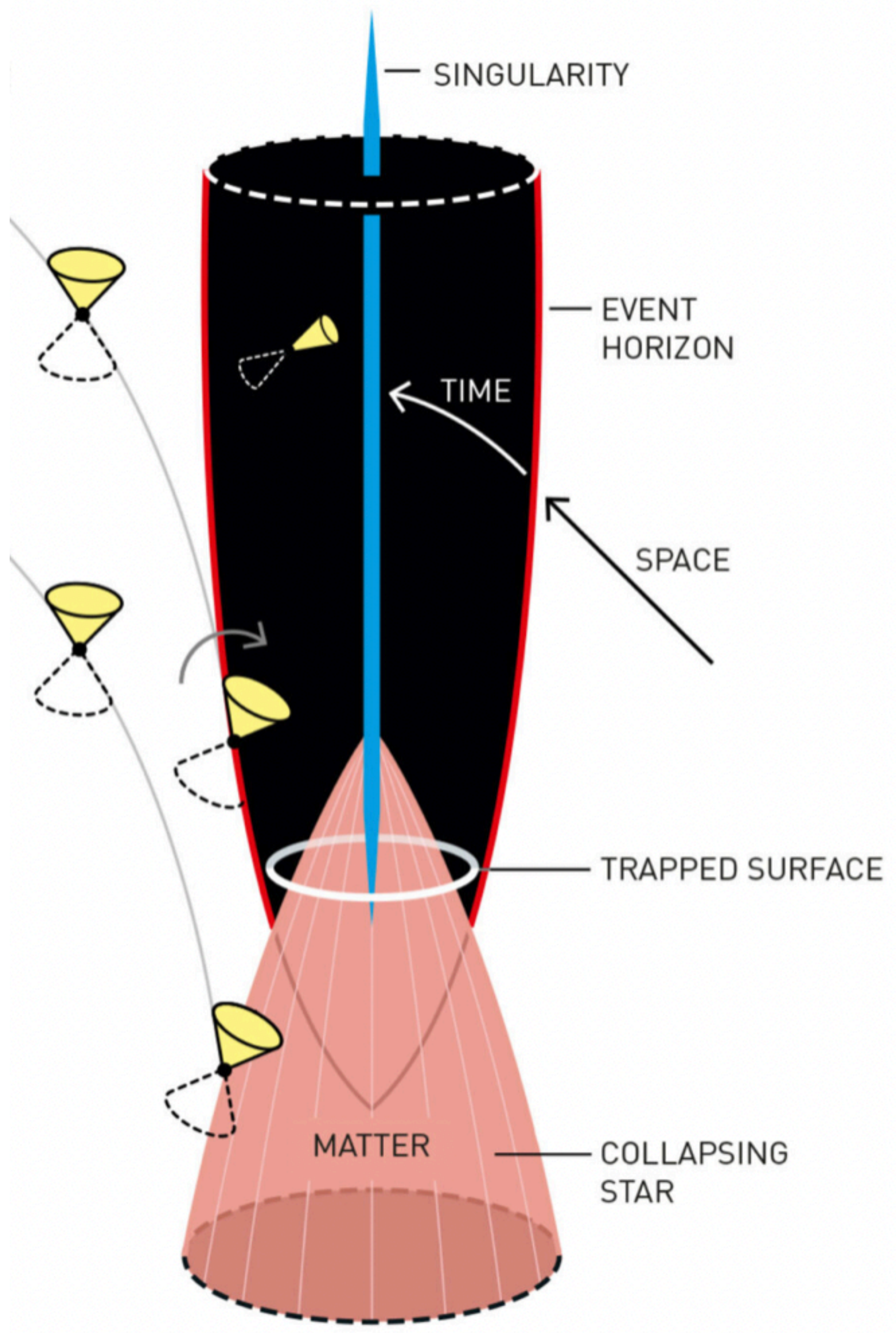
Outline

- *Weak and Strong Cosmic Censorship Conjecture*
- *Sorce-Wald's formulation for checking WCCC*
- *WCCC for extremal BH of Higher Derivative Theories*
- *WCCC for near-extremal BH of HDTs*
- *Violation of WCCC and Consistent Check*

Singularity & Cosmic Censorship



Singularity theorem Penrose-Hawking 1965

In general relativity, a singularity at which the spacetime ends is inevitable.

Cosmic Censorship Penrose 1969

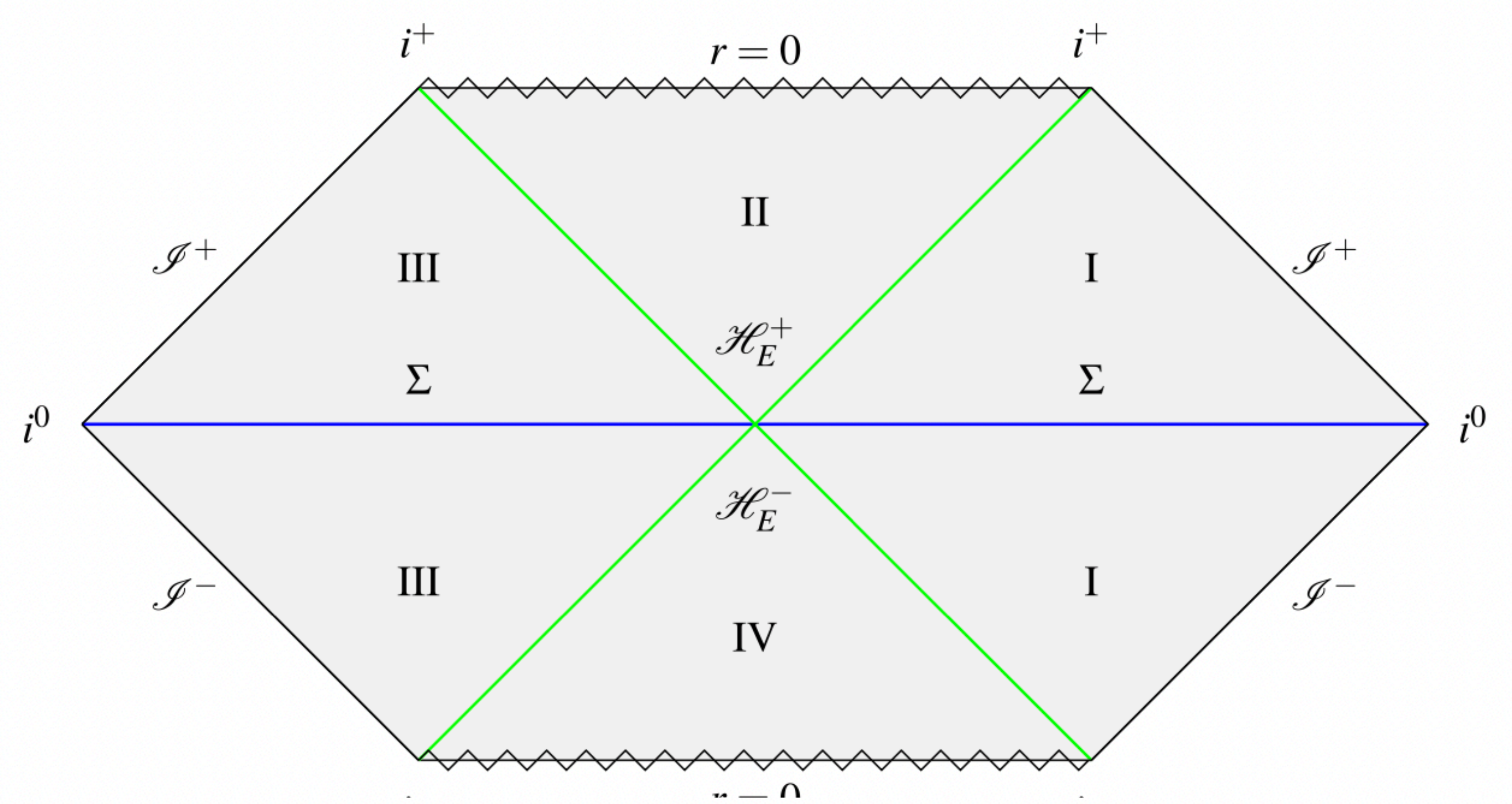
The physical nature of the singularity is unknown.

Penrose conjectured the cosmic censorship to require no acausal or indeterministic effect caused by the singularity.

Weak and Strong Cosmic Censorship

Two versions of Cosmic Censorship

Mathematically, cosmic censorship requires
the Cauchy development (grey) is globally hyperbolic.



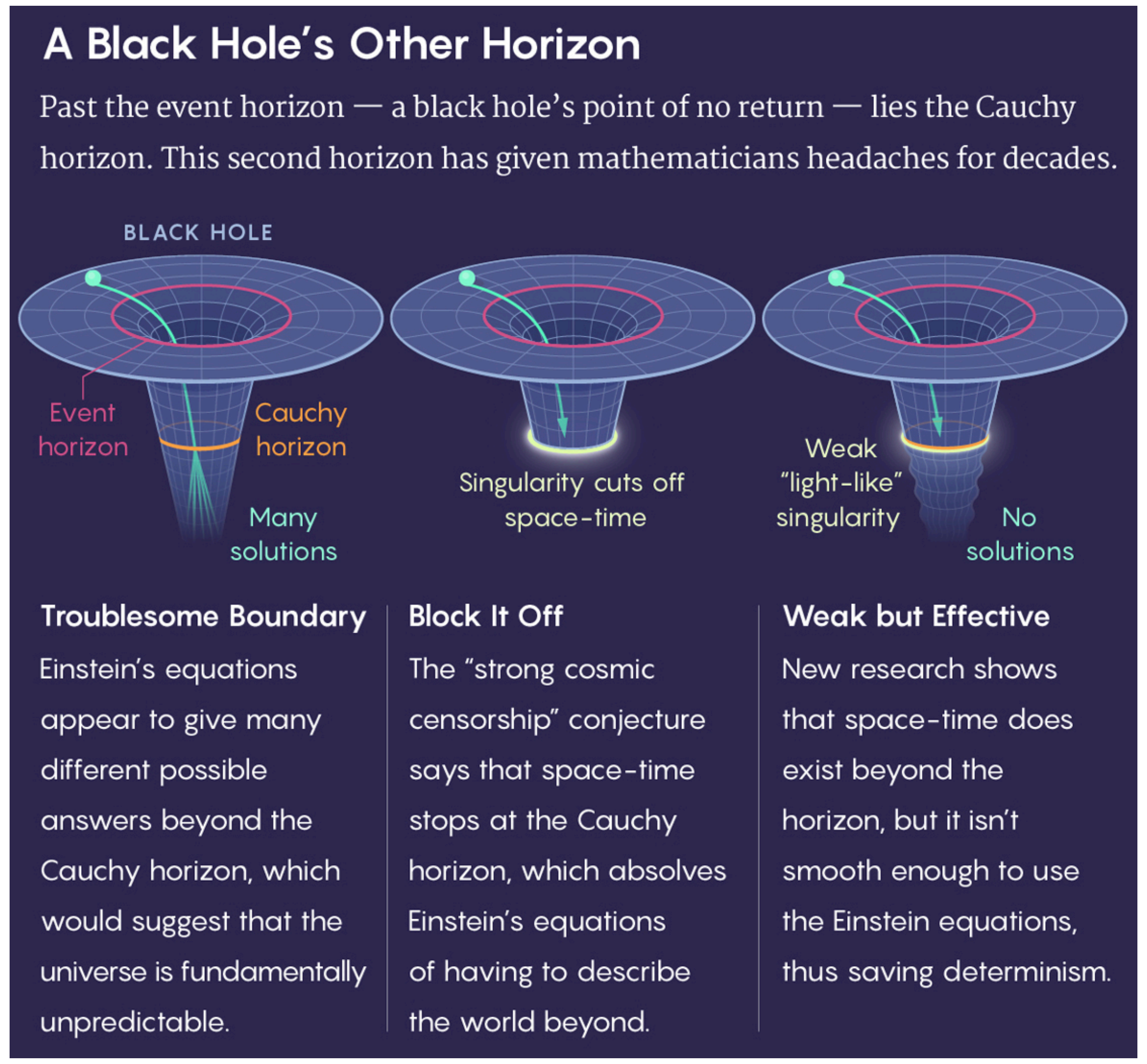
Weak versions

All singularity should be hidden by the event horizons (green), which are stable. I.e., no naked singularity.

Strong versions

It requires the instability
and ensuing disappearance
of Cauchy horizons (red).
I.e., inner horizon is unstable.

c.f. counterexamples found in
Dafermos & Luk 2017



A Black Hole's Other Horizon

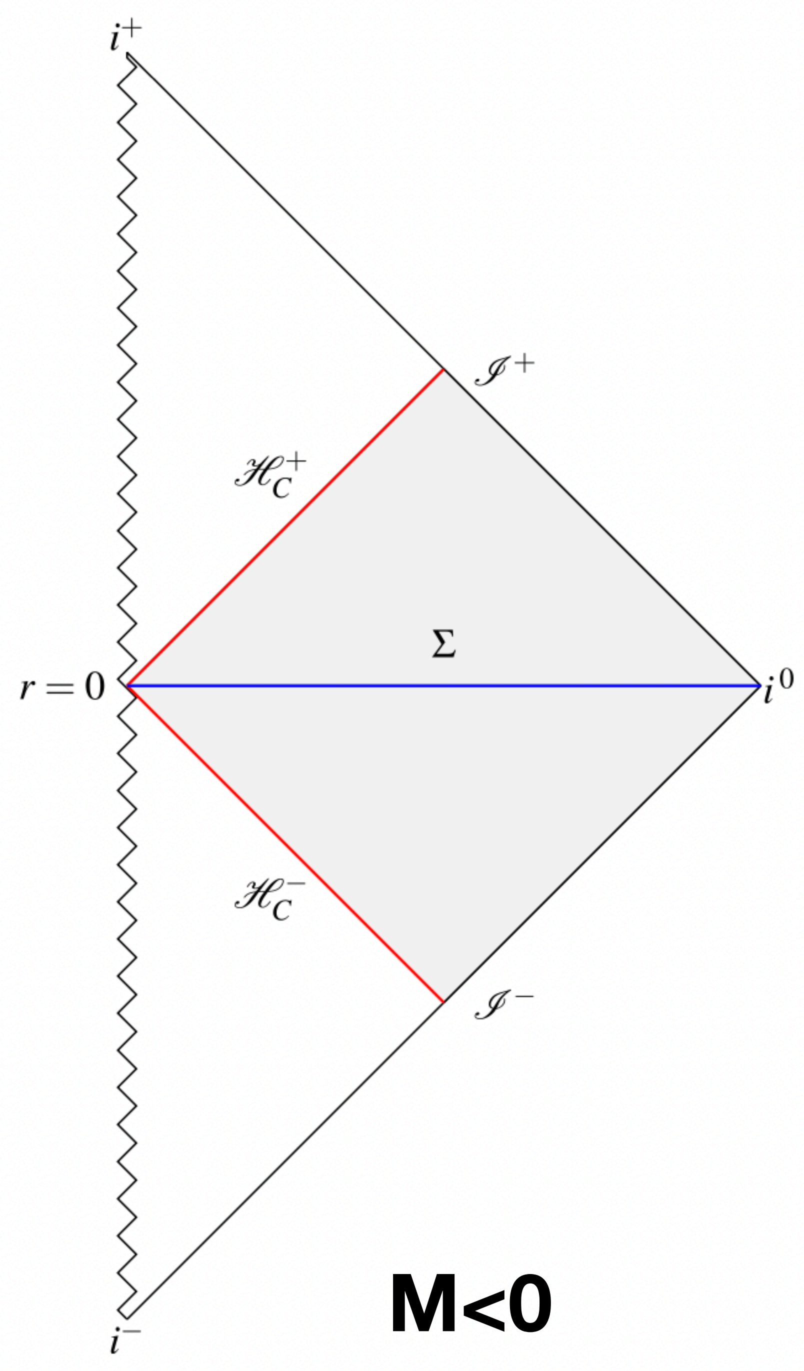
Past the event horizon — a black hole's point of no return — lies the Cauchy horizon. This second horizon has given mathematicians headaches for decades.

Troublesome Boundary
Einstein's equations appear to give many different possible answers beyond the Cauchy horizon, which would suggest that the universe is fundamentally unpredictable.

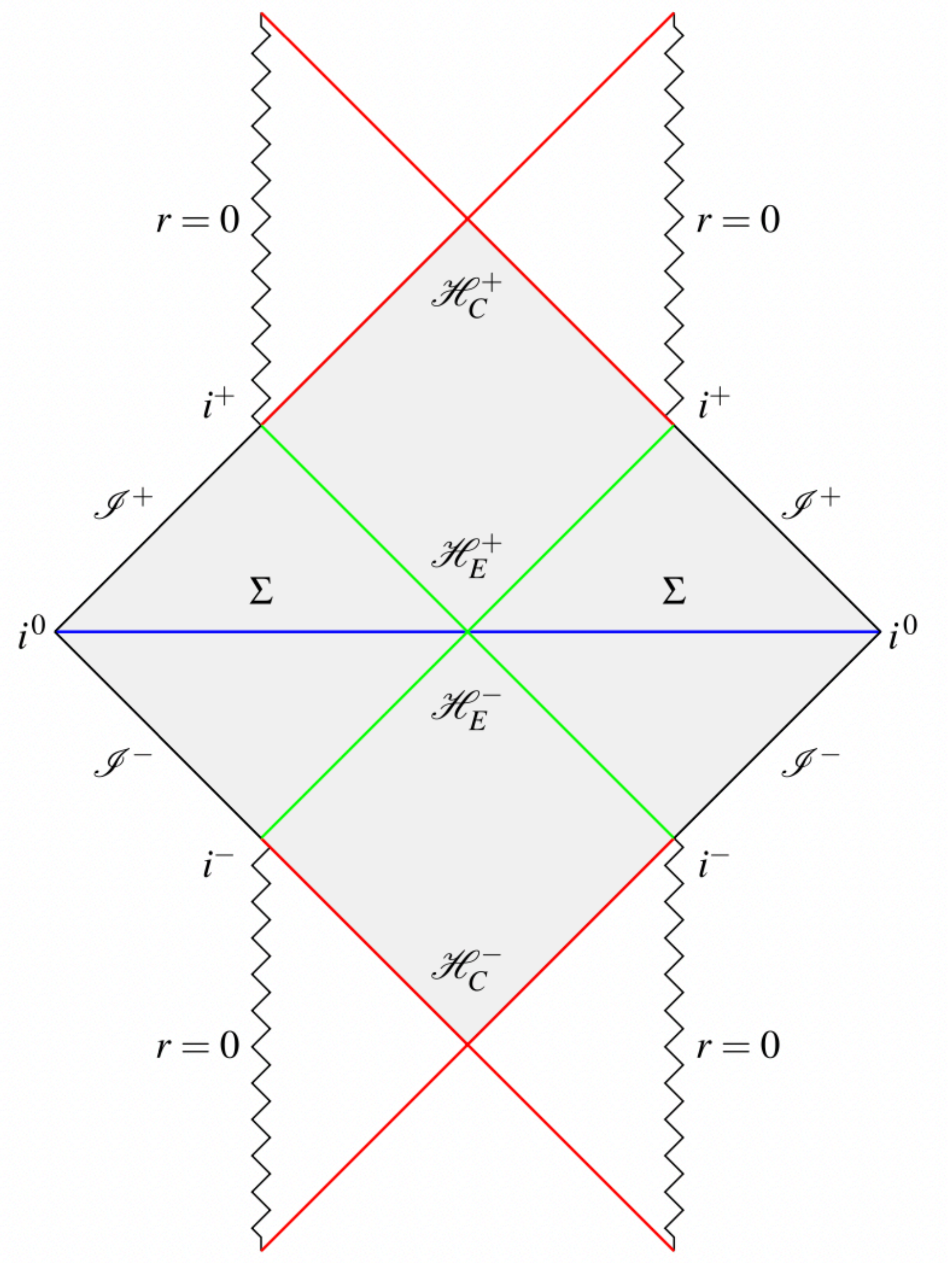
Block It Off
The "strong cosmic censorship" conjecture says that space-time stops at the Cauchy horizon, which absolves Einstein's equations of having to describe the world beyond.

Weak but Effective
New research shows that space-time does exist beyond the horizon, but it isn't smooth enough to use the Einstein equations, thus saving determinism.

c.f. Quanta Magazine



M<0
Schwarzschild



Kerr or RN

Wald's Gedanken Experiment

Wald (1974) gave the operational statement of WCCC by following gedanken experiment.

Throw the matter into (near-)extremal BH, then WCCC holds if energy condition holds.

$$m = E_\infty \underset{\substack{\uparrow \\ \textbf{energy conservation}}}{=} E_{\mathscr{H}} = -(mu\cdot\xi + qA\cdot\xi)|_{\mathscr{H}} \underset{\substack{\uparrow \\ \textbf{energy condition}}}{\geq} -qA\cdot\xi$$

For extremal RN BH, the electromagnetic potential on the horizon $\Phi_H = -A\cdot\xi = 1$,

$\therefore m \geq q\,.$ Thus, $M + m \geq Q + q\,.$

Some Issues for Wald's Gedanken Experiment

- Motion of the matter causes metric perturbation, which acts on the matter as self-force, and further induces radiation-reaction effect.

- The self-force is 2nd order effect, and will not affect the earlier analysis for extremal BH but the near-extremal BH.

- *Hubeny 1999* A near extremal BH with $\epsilon = \sqrt{1 - Q^2/M^2} \ll 1$ with $\Phi_H = \frac{Q}{r_+} = \frac{Q}{M(1+\epsilon)} \simeq 1 - \epsilon.$ Energy conservation and energy condition give $m \geq (1-\epsilon)q$.

- $M + m - (Q + q) \simeq -\epsilon q + M\epsilon^2/2.$ It seems that WCCC can be violated by taking $q > M\epsilon/2.$ This is not true because it neglects the self-force effect at $\mathcal{O}(q^2)$.

Sorce-Wald 2017

- Sorce & Wald develop a proof/check of WCCC by throwing generic matter into a (near-)extremal BH in Wald’s gedanken experiment.
- The proof/check is based on the energetic constraint without explicitly solving the real dynamics involving 2nd order self-force.
- The energetic constraint is derived from the Iyer-Wald formulation defining the covariant Noether charge & black hole mechanics/thermodynamics.
- Sorce & Wald use their formalism to prove WCCC for (near-)extremal BH of Einstein-Maxwell theory. We use this formalism to check WCCC for their higher derivative extensions.

Sorce-Wald 2017

Iyer-Wald formulation： ***covariant formulation of BH mechanics***

$\delta L = E(\phi) + d\Theta(\phi, \delta\phi),$ $\quad \phi = (g_{\mu\nu}, A_\mu),$ $\quad L =$ **Lagrangian 4-form**, $\quad E =$ **EoM**, $\quad \Theta =$ **symplectic 3-form**

1. Define Noether current given a vector ξ^μ: $J_\xi = \Theta(\phi, \mathscr{L}_\xi\phi) - i_\xi L$. It is easy to see $dJ_\xi = 0$ so that $J_\xi = dQ_\xi + \xi^\mu C_\mu$ with $(C_\mu)_{\alpha\beta\gamma} = \epsilon_{\nu\alpha\beta\gamma}(T^\nu_\mu + j^\nu A_\mu)$ where $T_{\mu\nu} \equiv (EoM)^g$, $J^\mu = (EoM)^A$.

2. $\delta J_\xi = di_\xi\Theta(\phi, \delta\phi)$ if $\mathscr{L}_\xi\phi = 0$. Together with $\delta J_\xi = d\delta Q_\xi + \xi^\mu\delta C_\mu$, this lead to the linear energetic constraint for BH when throwing into BH the matter obeying null energy condition:

$$\delta M - \Phi_H\delta Q = -\int_{\mathscr{H}} \epsilon_{\mu;3}\, \xi^\nu \delta T^\mu_\nu = 4\int_{\mathscr{H}} \epsilon_3\, \delta T_{\mu\nu} n^\mu n^\nu \geq 0.$$

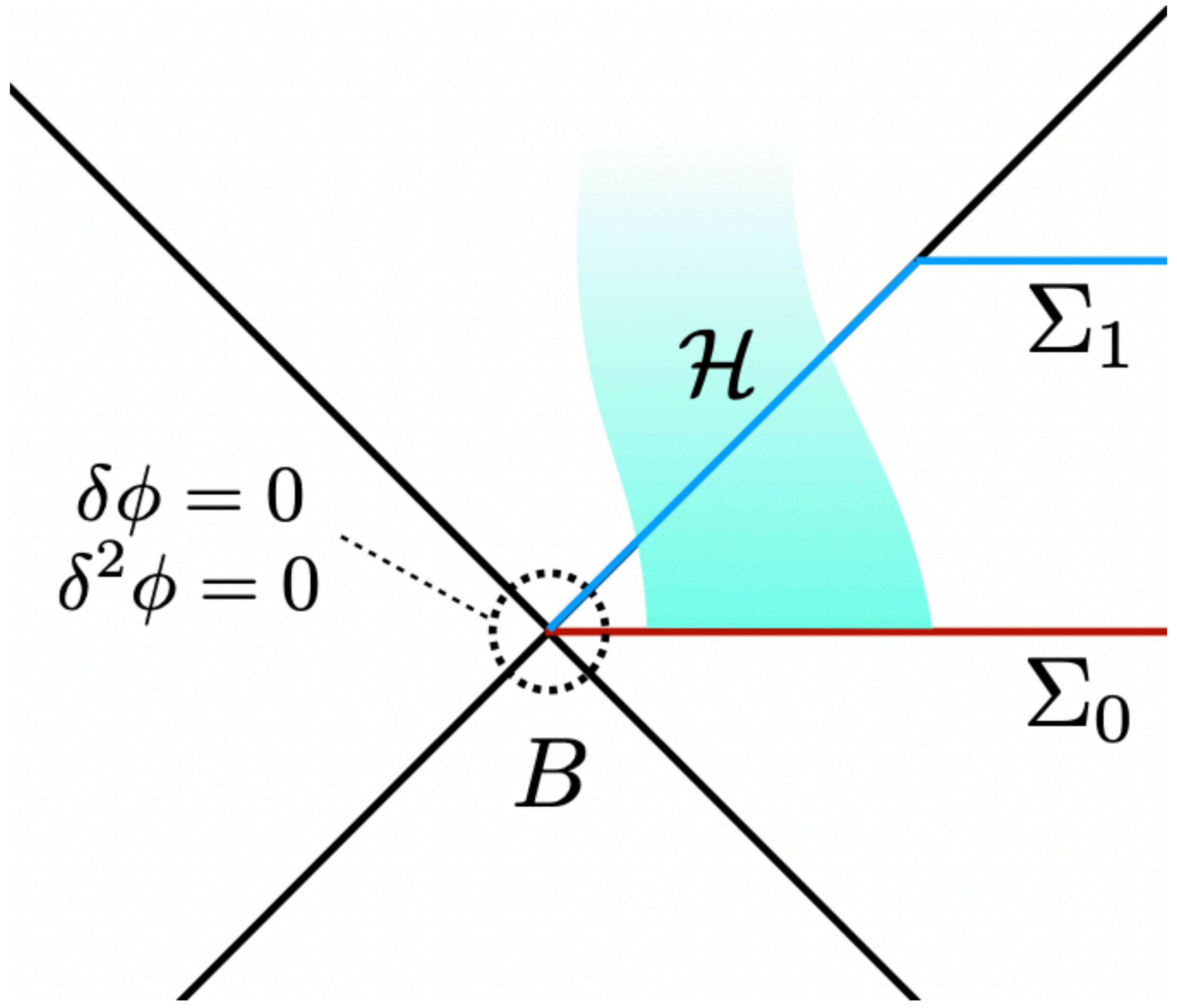
C.f. $\delta C_\mu = \epsilon_{\nu;3}(\delta T^\nu_\mu + A_\mu \delta j^\nu)$, $\quad \delta M \equiv \int_\infty [\delta Q_\xi - i_\xi\Theta(\phi, \delta\phi)]$, $\quad \delta Q \equiv \int_{\mathscr{H}} \epsilon_{\mu;3}\, \delta j^\mu$, $\quad \Phi_H \equiv -\xi^\mu A_\mu|_{\mathscr{H}}$.

Sorce-Wald 2017

A second variation of the linear energetic constraint gives the 2nd order energetic constraint:

$$\delta^2 M - \Phi_H \delta^2 Q = \mathscr{E}_\Sigma(\phi, \delta\phi) - \int_{\mathscr{H}} \epsilon_{\mu;3}\, \xi^\nu \delta^2 T^\mu_\nu \underset{\uparrow}{\geq} \mathscr{E}_\Sigma(\phi, \delta\phi)$$

null energy condition



c.f. Wald's canonical energy $\mathscr{E}_\Sigma(\phi, \delta\phi) = \int_{\Sigma = \mathscr{H} + \Sigma_1} \omega(\phi, \delta\phi, \mathscr{L}_\xi \phi) = \mathscr{E}_{\mathscr{H}} + \mathscr{E}_{\Sigma_1}$

with $\omega(\phi, \delta_1\phi, \delta_2\phi) = \delta_1 \Theta(\phi, \delta_2\phi) - \delta_2 \Theta(\phi, \delta_1\phi)$

$\mathscr{E}_{\mathscr{H}} \sim$ energy flux of gravitational & electromagnetic waves into BH ≥ 0 (??).

Assume $\delta\phi\,|_{\Sigma_1} = \delta\phi^{BH}$ s.t. $\delta^2 M = \delta^2 Q = \delta^2 C_\mu = 0$ and $\mathscr{E}_{\Sigma_1}(\phi, \delta\phi^{BH}) = \mathscr{E}_\Sigma(\phi, \delta\phi^{BH}) = -T_H \delta^2 S_{BH}$.

Thus, 2nd order energetic constraint takes the form of generalized 2nd law:

$$\delta^2 S_{BH} + \frac{1}{T_H}(\delta^2 M - \Phi_H \delta^2 Q) \geq 0$$

at least for collapsing spherical-shell of matter.

Short Summary of Overcharging a BH

Variate the extremality condition to obtain WCCC condition,

e.g., linear order WCCC condition for Einstein-Maxwell theory: $\delta M \geq \delta Q$.

To overcharge an extremal BH

Check the compatibility between WCCC condition and linear energetic constraint $\delta M - \Phi_H \delta Q \geq 0$. *E.g., for Einstein-Maxwell theory, the WCCC holds trivially.*

To overcharge a near-extremal BH

Assume the linear energetic constraint is saturated, i.e., , and use it and the 2nd order energetic constraint $\delta^2 M - \Phi_H \delta^2 Q \geq -T_H \delta^2 S_{BH}$ *to check if the 2nd order WCCC condition holds.*

Higher Derivative Theories

- The higher derivative extension of Einstein-Maxwell theory is inevitable by due to the loop correction of scalar and fermions, e.g., at 1-loop

$$L_{\text{spinor}} \propto 5RF^2 - 26R_{\mu\nu}F^{\mu\rho}F^{\nu}{}_{\rho} + 2R_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}\,,$$
$$L_{\text{scalar}} \propto -\frac{5}{2}RF^2 - 2R_{\mu\nu}F^{\mu\rho}F^{\nu}{}_{\rho} - 2R_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}\,,$$

- In this work, we will consider the following HDTs: $I = \int d^4x\ \sqrt{-g}(\frac{1}{2\kappa}R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \Delta L)\,,$

$$\begin{aligned}\Delta L = \quad & c_1R^2 + c_2R_{\mu\nu}R^{\mu\nu} + c_3R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \\ & + c_4\kappa RF_{\mu\nu}F^{\mu\nu} + c_5\kappa R_{\mu\nu}F^{\mu\rho}F^{\nu}{}_{\rho} + c_6\kappa R_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma} \\ & + c_7\kappa^2F_{\mu\nu}F^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} + c_8\kappa^2F_{\mu\nu}F^{\nu\rho}F_{\rho\sigma}F^{\sigma\mu}\,.\end{aligned}$$

- These theories can be tested by high energy experiments or gravitational wave observations.

BHs of HDTs *Kats et al 2006*

BH configuration

$$A_t = -\frac{q}{r} - \frac{\kappa^2 q^3}{5r^5}\left[c_2 + 4c_3 + 10c_4 + c_5 - c_6\kappa\left(9 - \frac{10mr}{q^2}\right) - 16c_7 - 8c_8\right] + \mathcal{O}(c_i^2)$$

$$\begin{aligned}-g_{tt} =& 1 - \frac{\kappa m}{r} + \frac{\kappa q^2}{2r^2} + c_2\left(\frac{\kappa^3 m q^2}{r^5} - \frac{\kappa^3 q^4}{5r^6} - \frac{2\kappa^2 q^2}{r^4}\right) + c_3\left(\frac{4\kappa^3 m q^2}{r^5} - \frac{4\kappa^3 q^4}{5r^6} - \frac{8\kappa^2 q^2}{r^4}\right) + c_4\left(-\frac{6\kappa^3 m q^2}{r^5} + \frac{4\kappa^3 q^4}{r^6} + \frac{4\kappa^2 q^2}{r^4}\right) \\ &+ c_5\left(-\frac{\kappa^3 m q^2}{r^5} + \frac{4\kappa^3 q^4}{5r^6}\right) + c_6\left(\frac{\kappa^3 m q^2}{r^5} - \frac{\kappa^3 q^4}{5r^6} - \frac{2\kappa^2 q^2}{r^4}\right) + c_7\left(-\frac{4\kappa^3 q^4}{5r^6}\right) + c_8\left(-\frac{2\kappa^3 q^4}{5r^6}\right) + \mathcal{O}(c_i^2)\,.\end{aligned}$$

extreamlity = double root of g_{tt}

$$m \geq \sqrt{\frac{2}{\kappa}}|q|\left(1 - \frac{4}{5q^2}c_0\right) \qquad c_0 \equiv c_2 + 4c_3 + c_5 + c_6 + 4c_7 + 2c_8$$

Weak Gravity Conjecture *requires $m/|q| < 1$, i.e., $c_0 > 0$ so that the number of stable particles is finite.*

WCCC for extremal BH

Variate the extremality condition gives

$$\delta m \geq \sqrt{\frac{2}{\kappa}}\left(1+\frac{4c_0}{5q^2}\right)\delta q \quad \longleftarrow \quad m \geq \sqrt{\frac{2}{\kappa}}|q|\left(1-\frac{4}{5q^2}c_0\right)$$

Wald's Energetic constraint

$$\delta m \geq \Phi_H \delta q \quad \text{with} \quad \Phi_H = -\xi \cdot A\,|_{\mathscr{H}} = \sqrt{\frac{2}{\kappa}}\left(1+\frac{4c_0}{5q^2}\right)$$

c.f. δm & δq receive no correction
from the higher derivative terms

→ *WCCC holds for extremal BH!*

WCCC for extremal BH

WCCC = Non-decreasing A_H or S_{BH}

(1) Assume $F(m, q, A_H) = 0$. Then, WCCC $\delta A_H = 0$ implies

$$\delta m = -\left(\frac{\partial_q F}{\partial_m F}\right)_S \delta q$$

(2) the extremality condition $\partial_A F(,mq, A_H) = 0$ implies

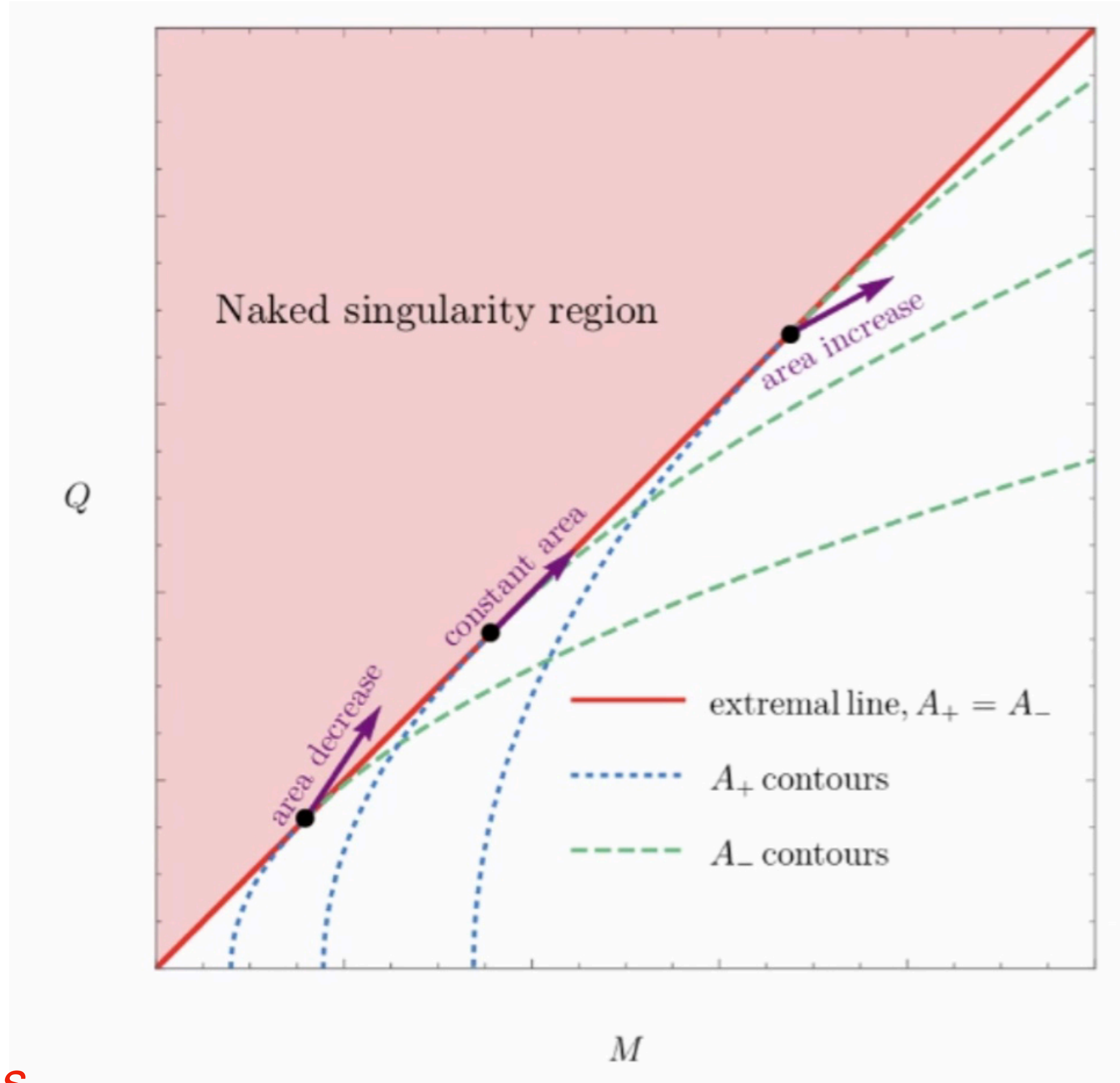
$$\left(\frac{\partial_q F}{\partial_m F}\right)_S = -\left(\frac{\partial m}{\partial q}\right)_{ext}$$

(1)+(2) gives $\delta m = \left(\frac{\partial m}{\partial q}\right)_{ext} \delta q$

First Law

$$dm = TdS_{BH} + \Phi_H dq \quad \xrightarrow{T \to 0} \quad \left(\frac{\partial m}{\partial q}\right)_{ext} = \Phi_H$$

Thus, WCCC requires $\delta m = \Phi_H \delta q$, which holds by Wald's linear energetic constraint for any gravity theory.



Extremality contour & constant area contours

Wald Entropy for HDTs

$$S_{BH} = -2\pi A_H \frac{\delta \mathscr{L}}{\delta R_{\mu\nu\rho\sigma}} \epsilon_{\mu\nu}\epsilon_{\rho\sigma}\Big|_{g_{\mu\nu},A_\mu,r_H} \Longrightarrow \quad -2\pi A_H \left[-\frac{1}{\kappa} - 4c_1 R - 4c_2 R^{rv} + 8c_3 R^{rvrv} + 2\kappa\,(2c_4 + c5 + 2c_6) F^{rv}F^{rv} \right]\Big|_{g_{\mu\nu},A_\mu,r_H}$$

RHS of 2nd order energetic constraint

$$\begin{aligned} T_H \delta^2 S_{BH} \;=\; & -\frac{1}{\epsilon^2 m}\Big[(1-2\epsilon)(\delta m - \delta q)^2 - 3\epsilon^2(\delta m - \delta q)\delta q + \epsilon^3(2\delta m - 3\delta q)\delta q\Big] + \frac{4c_2}{5\epsilon^2 m^3}\Big[\epsilon(14 - 74\epsilon + 217\epsilon^2)(\delta q)^2 \\ & + (2 - 32\epsilon + 139\epsilon^2 - 360\epsilon^3)\delta q \delta m + 2(1 - 9\epsilon + 32\epsilon^2 - 72\epsilon^3)(\delta m)^2\Big] + O(c_3, c_4, \cdots, c_8) \end{aligned}$$

seemingly singular

Apply linear energetic constraint

$$\delta m = \left[(1-\epsilon) + \frac{4}{5m^2}\Big(c_0(1+2\epsilon) + 10c_6\epsilon\Big)\right]\delta q$$

$$T_H \delta^2 S_{BH} \;=\; -\frac{1}{m}\left[1 - \frac{16}{5m^2}(2c_0 + 5c_6)\right](\delta q)^2 \,.$$

WCCC constraint

2nd order energetic constraint

$$\delta^2 m - \Phi_H \delta^2 q \geq -T_H \delta^2 S_{BH} \xrightarrow[\text{opetimal linear energetics}]{\text{Wald Entropy}} \delta^2 m \geq \left[1 + \frac{4c_0}{5m^2}\right]\delta^2 q + \frac{1}{m}\left[1 - \frac{16}{5m^2}(2c_0 + 5c_6)\right](\delta q)^2$$

Extremality condition

Expand $f(\lambda) = m^2(\lambda) - q^2(\lambda)\left(1 - \frac{4c_0}{5q^2(\lambda)}\right)^2$ up to 2nd order by $m(\lambda) = m + \lambda\delta m + \frac{\lambda^2}{2}\delta^2 m$ & $q(\lambda) = q + \lambda\delta q + \frac{\lambda^2}{2}\delta^2 q$

$\Big\downarrow$ 2nd order energetic constraint

$$f(\lambda) \;=\; (\epsilon m - \lambda\delta q)^2 + \frac{8}{5m^2}(\epsilon m - \lambda\delta q)\Big(c_0(\epsilon m + 3\lambda\delta q) + 10c_6\lambda\delta q\Big)\,.$$

$$c_0 \equiv c_2 + 4c_3 + c_5 + c_6 + 4c_7 + 2c_8$$

Violation of WCCC

WCCC constraint

$$f(\lambda) = (\epsilon m - \lambda\delta q)^2 + \frac{8}{5m^2}(\epsilon m - \lambda\delta q)\Big(c_0(\epsilon m + 3\lambda\delta q) + 10c_6\lambda\delta q\Big)\,.$$

$$c_0 \equiv c_2 + 4c_3 + c_5 + c_6 + 4c_7 + 2c_8$$

Note 1: No $\delta^2 m$ and $\delta^2 q$ appears. The leading complete-square term is the one of Einstein-Maxwell theory as expected.

Note 2: WCCC is always preserved if $c_0 = c_6 = 0$. This is different from the constraint $c_0 > 0$ by weak gravity conjecture.

Note 3: No clue why c_6 is exceptional.

Assume $\lambda \sim \epsilon \ll c_i \ll 1$ *and* $\lambda\delta q \gtrsim \epsilon m > 0$ *s.t.* $|\epsilon m - \lambda\delta q| \approx \frac{d_1}{m} \ll 1$ *for some* $d_1 > 0$. *Then,*

$$f(\lambda) \approx \frac{d_1^2}{m^2}\left(1 - \frac{16}{d_1}\epsilon(2c_0 + 5c_6)\right)$$ *so that WCCC can be violated if* $\epsilon(2c_0 + 5c_6) > \frac{d_1}{16}$.

This can be achieved easily.

Spherical Thin-Shell in EGB gravity

According to WCCC constraint, WCCC is preserved for Einstein-Gauss-Bonnet (EGB) gravity, i.e., $c_1 = c_3 = -\frac{1}{4}c_2 \equiv c_{GB}$. Its black hole solution is just the same as Einstein-Maxwell.

In this case, the junction condition is of first order and we can consider a spherical thin-shell for a consistent check of the WCCC constraint.

Thin-shell junction condition for EGB metric $ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega$

$\left[K_{\mu\nu} - h_{\mu\nu}K + 2c_{GB}(3J_{\mu\nu} - h_{\mu\nu}J + 2\hat{P}_{\mu\rho\lambda\nu}K^{\rho\lambda})\right]_J = -S_{\mu\nu}$ *with* $J_{\mu\nu} = \frac{1}{3}(2KK_{\mu\rho}K^{\rho}_{\nu} + K_{\rho\lambda}K^{\rho\lambda}K_{\mu\nu} - 2K_{\mu\rho}K^{\rho\lambda}K_{\lambda\nu} - K^2K_{\mu\nu}) \neq 0$

and $\hat{P}_{\mu\nu\rho\lambda} = \hat{R}_{\mu\nu\rho\lambda} + 2\hat{R}_{\nu[\rho}h_{\lambda]\mu} - 2\hat{R}_{\mu[\rho}h_{\lambda]\nu} + \hat{R}_{\nu[\rho}h_{\lambda]\mu} + h_{\mu[\rho}h_{\lambda]\nu}\hat{R}$ *where the hatted is evaluated w.r.t. induced metric* $h_{\mu\nu}$.

Straightforwardly to find $\hat{P}_{\mu\nu\rho\lambda} = 3J_{\mu\nu} - h_{\mu\nu}J = 0$ *so that the junction condition reduces to the Einstein-Maxwell one.*

Floating Thin-Shell

choose the metric on either sides to be $f_+(r) = \frac{1 - 2\frac{m_-}{r_s} + \frac{q_-^2}{r_s^2}}{1 - 2\frac{m_+}{r_s} + \frac{q_+^2}{r_s^2}} f_-(r) = 1 - 2\frac{m_+}{r} + \frac{q_+^2}{r^2}$ *for the metric to be continuous at the junction* $r = r_s$.

Assume the thin-shell matter is pressure-less, then the junction condition gives $m_+^2 - q_+^2 = \left(\frac{r_s - m_+}{r_s - m_-}\right)^2 (m_-^2 - q_-^2)$.

This is consistent with WCCC.

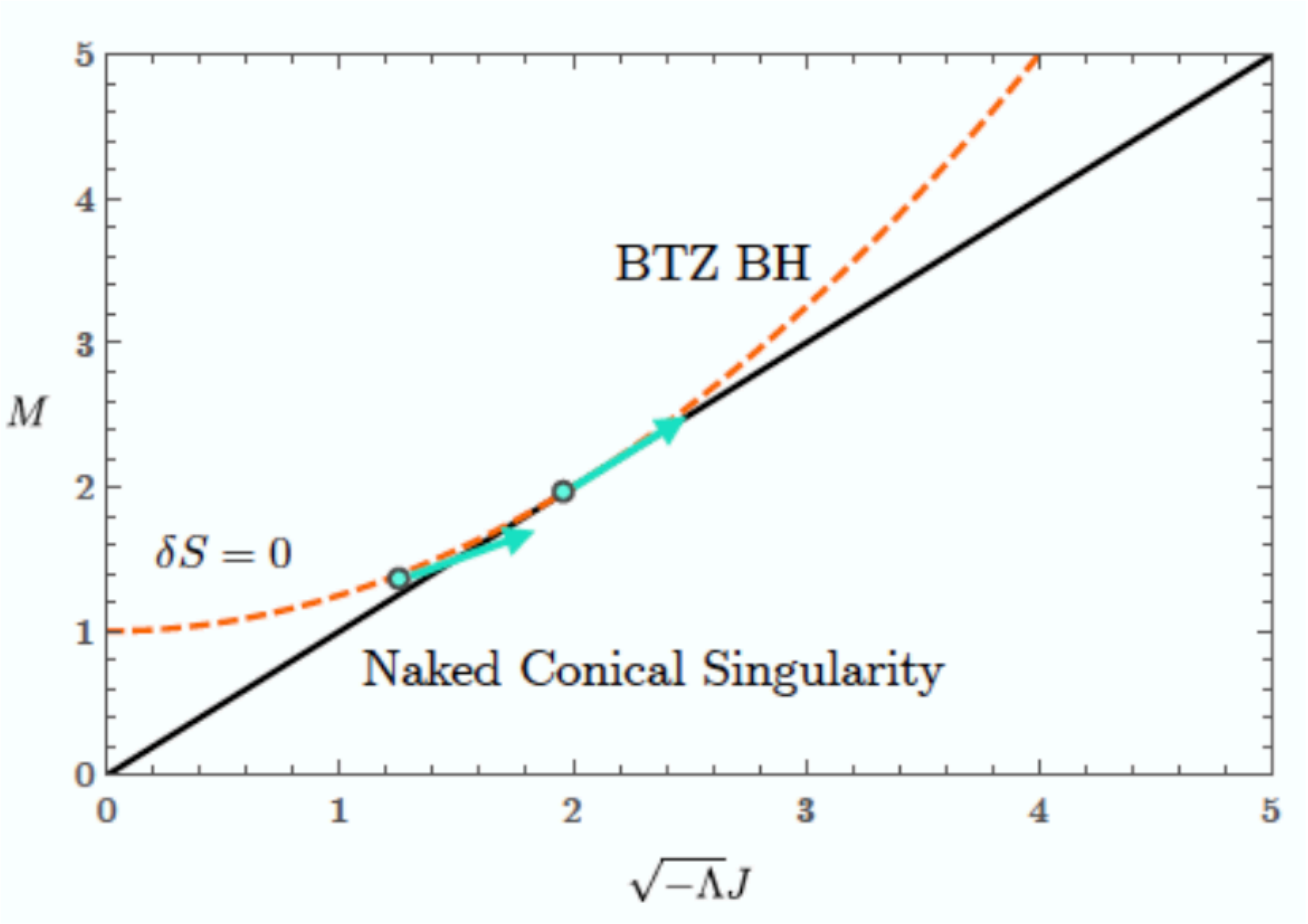
WCCC for BTZ BH of 3D Gravity

*We also check WCCC for the BTZ BH of 3D gravity theories for which the null energy condition is well-defined, (a) **3D Einstein gravity**; (b) **3D chiral gravity** which is of higher derivative. Both are torsion free.*

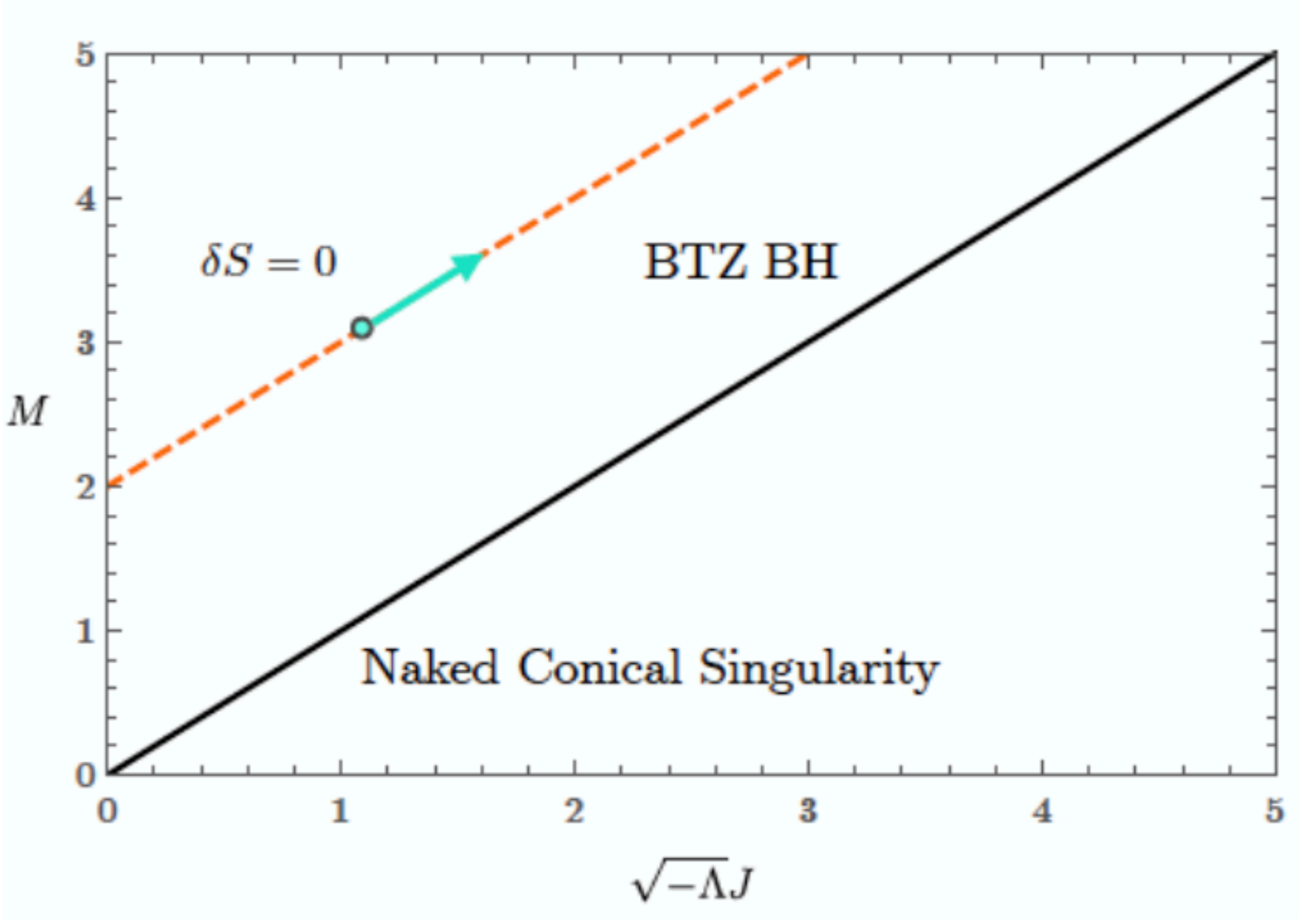
Apply Sorce-Wald, and we find that WCCC holds for both cases.

$$\because \; M = \left(\sqrt{-\Lambda}\right) J - \frac{\Lambda}{16\pi^2} S^2$$

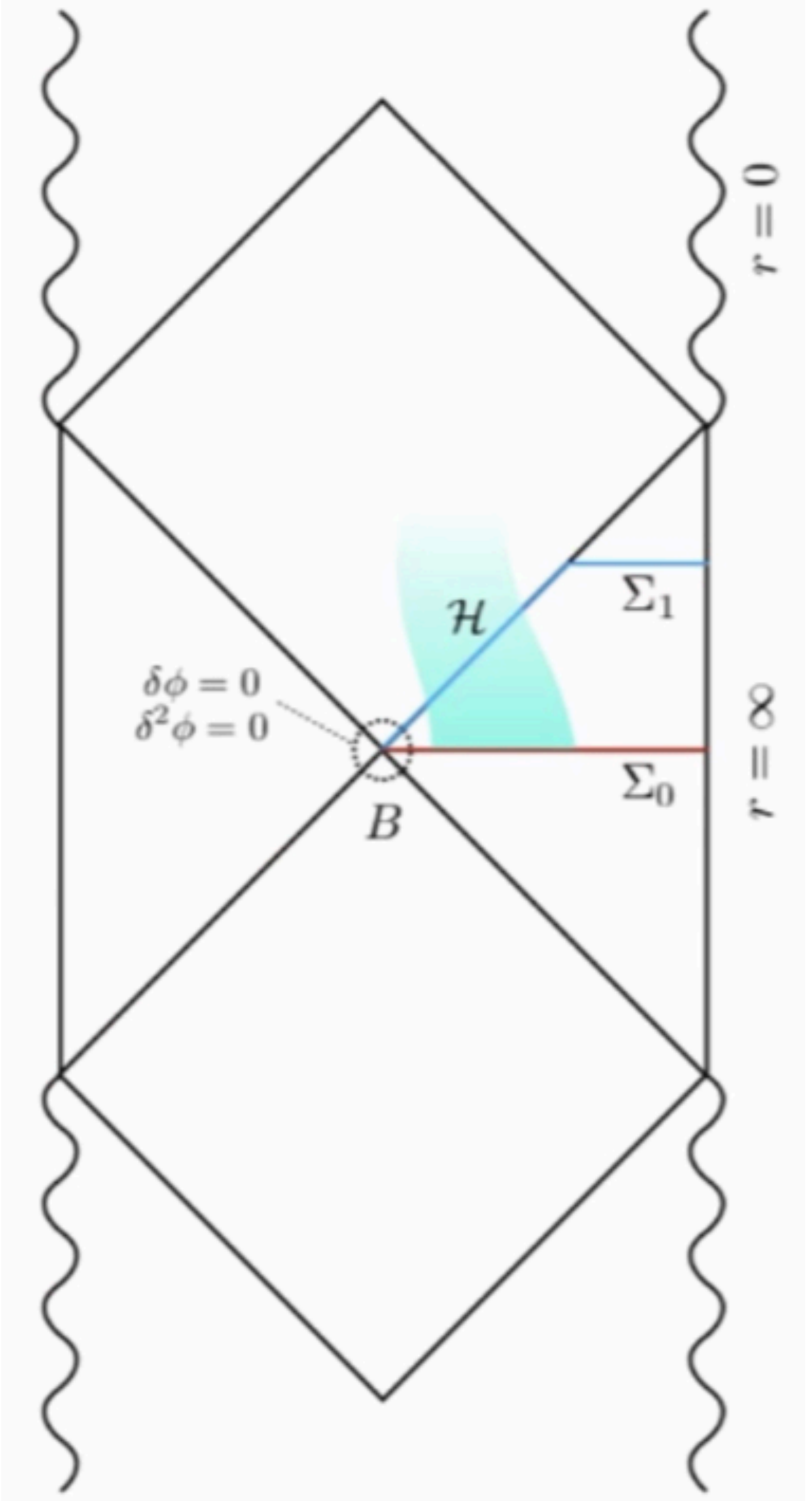
Optimal linear energetics implies $\delta S = 0$.



3D Einstein gravity



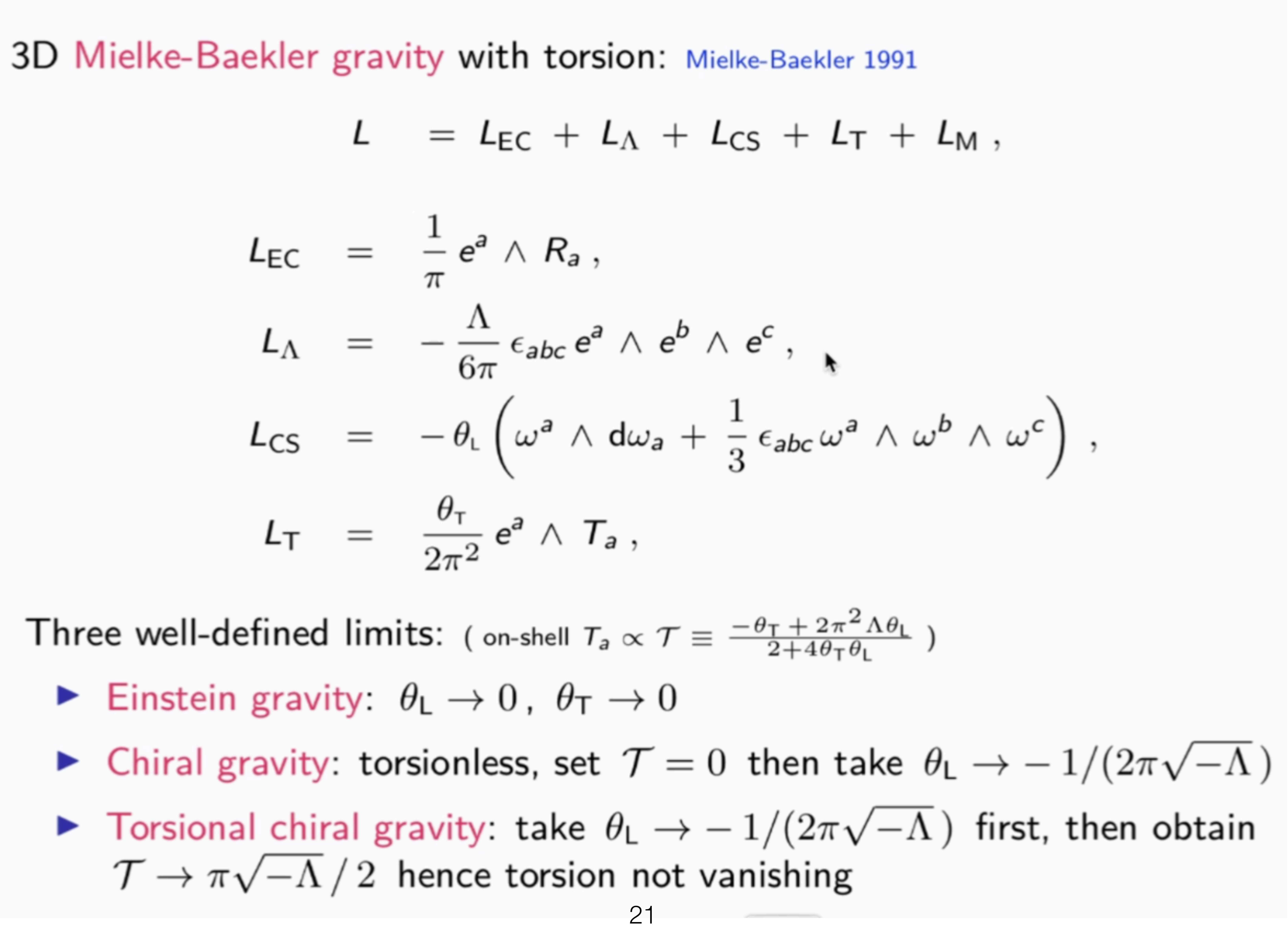
3D chiral gravity



Conclusion

- *Cosmic censorship is a fundamental issue in general relativity*
- *We find that WCCC holds for extremal black holes in generic theories of gravity.*
- *However, we find some evidence that WCCC can be violated for some higher derivative extension of Einstein gravity.*
- *Despite that, a direct example of WCCC violation is wanted.*
- *Our constraint can be relevant for UV completion as the one derived from weak gravity conjecture.*

Supplement I



3D Mielke-Baekler gravity with torsion: Mielke-Baekler 1991

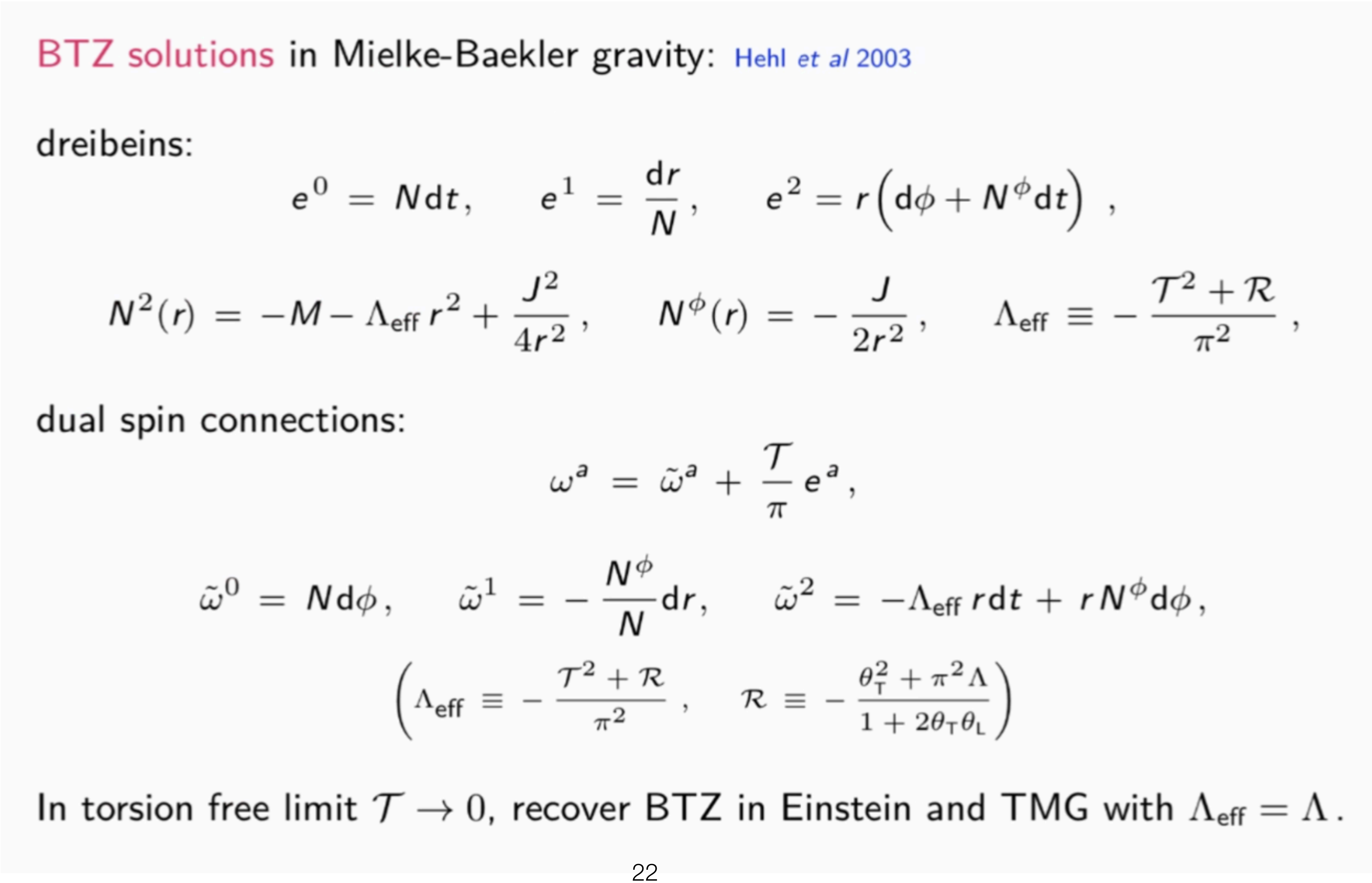
$$L = L_{\mathrm{EC}} + L_{\Lambda} + L_{\mathrm{CS}} + L_{\mathrm{T}} + L_{\mathrm{M}} \,,$$

$$\begin{aligned} L_{\mathrm{EC}} &= \frac{1}{\pi}\, e^a \wedge R_a \,, \\ L_{\Lambda} &= -\frac{\Lambda}{6\pi}\, \epsilon_{abc}\, e^a \wedge e^b \wedge e^c \,, \\ L_{\mathrm{CS}} &= -\theta_{\mathrm{L}} \left(\omega^a \wedge \mathrm{d}\omega_a + \frac{1}{3}\, \epsilon_{abc}\, \omega^a \wedge \omega^b \wedge \omega^c \right) \,, \\ L_{\mathrm{T}} &= \frac{\theta_{\mathrm{T}}}{2\pi^2}\, e^a \wedge T_a \,, \end{aligned}$$

Three well-defined limits: (on-shell $T_a \propto \mathcal{T} \equiv \frac{-\theta_{\mathrm{T}} + 2\pi^2 \Lambda \theta_{\mathrm{L}}}{2+4\theta_{\mathrm{T}}\theta_{\mathrm{L}}}$)

- Einstein gravity: $\theta_{\mathrm{L}} \to 0\,,\ \theta_{\mathrm{T}} \to 0$
- Chiral gravity: torsionless, set $\mathcal{T} = 0$ then take $\theta_{\mathrm{L}} \to -1/(2\pi\sqrt{-\Lambda}\,)$
- Torsional chiral gravity: take $\theta_{\mathrm{L}} \to -1/(2\pi\sqrt{-\Lambda}\,)$ first, then obtain $\mathcal{T} \to \pi\sqrt{-\Lambda}\,/\,2$ hence torsion not vanishing

Supplement II



BTZ solutions in Mielke-Baekler gravity: Hehl *et al* 2003

dreibeins:

$$e^0 = N\mathrm{d}t\,, \qquad e^1 = \frac{\mathrm{d}r}{N}\,, \qquad e^2 = r\Big(\mathrm{d}\phi + N^\phi \mathrm{d}t\Big)\ ,$$

$$N^2(r) = -M - \Lambda_{\text{eff}}\, r^2 + \frac{J^2}{4r^2}\,, \qquad N^\phi(r) = -\frac{J}{2r^2}\,, \qquad \Lambda_{\text{eff}} \equiv -\frac{\mathcal{T}^2 + \mathcal{R}}{\pi^2}\,,$$

dual spin connections:

$$\omega^a = \tilde{\omega}^a + \frac{\mathcal{T}}{\pi}\, e^a\,,$$

$$\tilde{\omega}^0 = N\mathrm{d}\phi\,, \qquad \tilde{\omega}^1 = -\frac{N^\phi}{N}\mathrm{d}r\,, \qquad \tilde{\omega}^2 = -\Lambda_{\text{eff}}\, r\mathrm{d}t + r N^\phi \mathrm{d}\phi\,,$$

$$\left(\Lambda_{\text{eff}} \equiv -\frac{\mathcal{T}^2 + \mathcal{R}}{\pi^2}\,, \qquad \mathcal{R} \equiv -\frac{\theta_{\mathrm{T}}^2 + \pi^2\Lambda}{1 + 2\theta_{\mathrm{T}}\theta_{\mathrm{L}}}\right)$$

In torsion free limit $\mathcal{T} \to 0$, recover BTZ in Einstein and TMG with $\Lambda_{\text{eff}} = \Lambda$.