Quantum black holes from matrix models

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Introduction

- Microscopic studies of black holes often demand exploring challenging quantum regimes (e.g. strong coupling, complex microstates, …)

- Solvable (but still nontrivial) models in a well-defined setup
- BPS black holes & quantitative lessons thereof.
- AdS/CFT & precise definition of quantum gravity.

- Today, we study the maximal SYM index & see how it views BH's in $AdS_5 \times S^5$.
- 2004: BPS black holes constructed. [Gutowski, Reall] ……
- 2005: Constructed the an index. [Kinney, Maldacena, Minwalla, Raju]

  ...
- 2018 ~ : Started to understand how to see black holes from this index.

- Still, no exact large N saddle point solutions known in dual QFT, except in special limits (small/large charges). Today I explain how to construct them.

The N=4 index on $S^3 \times R$

- Counts BPS states saturating $E \geq Q_1 + Q_2 + Q_3 + J_1 + J_2$

- $Q_{I=1,2,3}$ are Cartans of $SO(6)$ R-charges, $J_{i=1,2}$ those of $SO(4)$ on S^3.

$$Z(\Delta_I, \omega_i) = \mathrm{Tr}\left[(-1)^F e^{-\sum_{I=1}^{3} \Delta_I Q_I - \sum_{i=1}^{2} \omega_i J_i}\right]$$

$$\Delta_1 + \Delta_2 + \Delta_3 - \omega_1 - \omega_2 = 0 \quad \text{mod } 2\pi i\, Z$$

- Matrix integral representation for U(N) gauge group

$$Z(\delta_I, \sigma, \tau) \sim \frac{1}{N!} \prod_{a=1}^{N} \int_{-\frac{1}{2}}^{\frac{1}{2}} du_a \cdot \prod_{a \neq b} \frac{\prod_{I=1}^{3} \Gamma(\delta_I + u_{ab}, \sigma, \tau)}{\Gamma(u_{ab}, \sigma, \tau)}$$

$$u_{ab} \equiv u_a - u_b$$

$$\Delta_I = -2\pi i \delta_I \ , \quad \omega_1 = -2\pi i \sigma \ , \quad \omega_2 = -2\pi i \tau$$

- Fixing period conventions, take either $\delta_1 + \delta_2 + \delta_3 - \sigma - \tau = \pm 1$. (complex-conjugate sectors)

- Elliptic gamma function:

$$\Gamma(z, \sigma, \tau) \equiv \prod_{m,n=0}^{\infty} \frac{1 - e^{-2\pi i z} e^{2\pi i((m+1)\sigma + (n+1)\tau)}}{1 - e^{2\pi i z} e^{2\pi i(m\sigma + n\tau)}}$$

- Understanding its properties is the starting point of our construction.

Elliptic gamma function

- $SL(3,Z)$ modularity (on "$T^3 \sim (S^1)^3$" in $S^3 \times S^1$)

- "S-duality":
$$\Gamma(z,\sigma,\tau) = e^{-\pi i Q_+(z,\sigma,\tau)}\Gamma(\tfrac{z}{\tau},-\tfrac{1}{\tau},\tfrac{\sigma}{\tau})\Gamma(\tfrac{-z-1}{\sigma},-\tfrac{1}{\sigma},-\tfrac{\tau}{\sigma})$$
$$\Gamma(z,\sigma,\tau) = e^{-\pi i Q_-(z,\sigma,\tau)}\Gamma(-\tfrac{z}{\sigma},-\tfrac{1}{\sigma},-\tfrac{\tau}{\sigma})\Gamma(\tfrac{z-1}{\tau},-\tfrac{1}{\tau},\tfrac{\sigma}{\tau})$$
$$Q_\pm = \frac{z^3}{3\sigma\tau} - \frac{\sigma+\tau\mp 1}{2\sigma\tau}z^2 + \frac{\sigma^2+\tau^2+3\sigma\tau\mp 3\sigma\mp 3\tau+1}{6\sigma\tau}z + \frac{1}{12}(\sigma+\tau\mp 1)\left(\frac{1}{\sigma}+\frac{1}{\tau}\mp 1\right)$$

- Period: $\Gamma(z+1,\sigma,\tau) = \Gamma(z,\sigma,\tau)$

- Quasi-periods: $\Gamma(z+\sigma,\sigma,\tau) = \theta(z,\tau)\Gamma(z,\sigma,\tau)$, $\Gamma(z+\tau,\sigma,\tau) = \theta(z,\sigma)\Gamma(z,\sigma,\tau)$

(Similar to $SL(2,Z)$ on T^2: q-theta function $\theta(z/\tau,-1/\tau) = e^{\pi i B(z)}\theta(z,\tau)$, etc.)

- Assume $Im(\sigma/\tau) > 0$. "S-dual rewriting" of the integrand

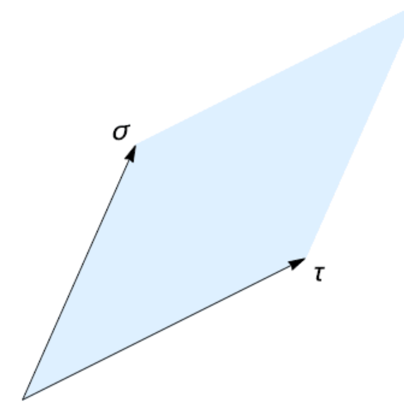
$$Z = \exp\left[-\frac{\pi i N^2\delta_1\delta_2\delta_3}{\sigma\tau}\right]\cdot\frac{1}{N!}\int d^N u \underbrace{\frac{\prod_{I=1}^3 \Gamma\left(-\frac{\delta_I+u_{ab}+1}{\sigma},-\frac{1}{\sigma},-\frac{\tau}{\sigma}\right)}{\Gamma\left(-\frac{u_{ab}+1}{\sigma},-\frac{1}{\sigma},-\frac{\tau}{\sigma}\right)}}\cdot\underbrace{\frac{\prod_{I=1}^3 \Gamma\left(\frac{\delta_I+u_{ab}}{\tau},-\frac{1}{\tau},\frac{\sigma}{\tau}\right)}{\Gamma\left(\frac{u_{ab}}{\tau},-\frac{1}{\tau},\frac{\sigma}{\tau}\right)}}$$

$\equiv I_\sigma(u) = e^{-V_\sigma(u)}$: periodic in $u_a \to u_a + \sigma$

$\equiv I_\tau(u) = e^{-V_\tau(u)}$: periodic in $u_a \to u_a + \tau$

- Remember → Quasi-periodicity realized as "factorization + exact σ,τ-periodicities"

Crude idea



- I first present semi-correct (thus semi-wrong) ideas.

- Factorized potential & separate periods:

  Uniform parallelogram distribution yields vanishing force.

$$u(x,y) \equiv \sigma x + \tau y \; , \; -1/2 < x, y < 1/2$$

$$\frac{\partial}{\partial u_2} = -\frac{\partial}{\partial u_1} = -\frac{\partial}{\sigma \partial x_1} = -\frac{\partial}{\tau \partial y_1}$$

$$\frac{\partial}{\partial u_2} \int_{-\frac{1}{2}}^{\frac{1}{2}} dx_1 dy_1 \left[V_\sigma(u_{12}) + V_\tau(u_{12}) \right] = - \int_{-\frac{1}{2}}^{\frac{1}{2}} dx_1 dy_1 \left[\frac{1}{\sigma} \frac{\partial V_\sigma(u_{12})}{\partial x_1} + \frac{1}{\tau} \frac{\partial V_\tau(u_{12})}{\partial y_1} \right] = 0$$

force on eigenvalue at u_2

$V_\sigma(\cdots + \sigma) - V_\sigma(\cdots) = 0$ $\quad$ $V_\tau(\cdots + \tau) - V_\tau(\cdots) = 0$

- Caveat: Taking log may yield branch cuts, spoiling periodicities.

- Branch points in the domain of eigenvalue distribution (multi-parallelogram)…?

- Always ∃ univeral branch points. "Haar measure singularity"

$$\int [dU] = \frac{1}{N!} \int d^N u \prod_{a \neq b} (1 - e^{2\pi i u_{ab}})$$

After "S-dual rewriting," singularity encoded in the following factor

$$\prod_{a \neq b} \left(1 - e^{\frac{2\pi i u_{ab}}{\tau}} \right)$$

Curing the caveat

- Haar measure singularity: Slightly reformulate the problem to evade it.

- If $f(u)$ is permutation-invariant, can replace Haar-like measure by half ("Molien-Weyl")

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} d^N u \cdot \frac{1}{N!} \prod_{a<b} (1-e^{2\pi i \kappa u_{ab}})(1-e^{-2\pi i \kappa u_{ab}}) \cdot f(u) = \int_{-\frac{1}{2}}^{\frac{1}{2}} d^N u \cdot \prod_{a<b} (1-e^{2\pi i \kappa u_{ab}}) \cdot f(u)$$

- Insert $\kappa = 1/\tau$ and order the eigenvalues properly in the multi-parallelogram.
- Can always have $\left|e^{2\pi i u_{ab}/\tau}\right| < 1$ for $a < b$: no Haar measure singularity on RHS.

- Other branch point singularities stay outside our domain if

$$\mathrm{Im}\left(\frac{\sigma-\delta_I}{\tau}\right) < 0\ , \quad \mathrm{Im}\left(\frac{1+\delta_I}{\tau}\right) < 0\ , \quad \mathrm{Im}\left(\frac{1-\tau+\delta_I}{\sigma}\right) < 0\ , \quad \mathrm{Im}\left(\frac{\delta_I}{\sigma}\right) > 0$$

- If this constraint is met, our ansatz is a saddle point of RHS.
- Looks stupid that branch points obstruct our way. But its has a physics interpretation.

Free energy and entropy

- Free energy (on one of the two surfaces $\Delta_1 + \Delta_2 + \Delta_3 - \omega_1 - \omega_2 = \pm 2\pi i$)

$$\log Z = -\frac{\pi i N^2 \delta_1 \delta_2 \delta_3}{\sigma \tau} = \frac{N^2 \Delta_1 \Delta_2 \Delta_3}{2\omega_1 \omega_2}$$

- This function was first discovered by studying BH solutions [Hosseini, Hristov, Zaffaroni] (2017).
- Having got the same function from QFT, it derives the Bekenstein-Hawking entropy of BH's.

- "Entropy": Legendre transformation on the surfaces yield complex functions.

$$S(Q_I, J_i; \Delta_I, \omega_i) = \log Z + \sum_I \Delta_I Q_I + \sum_i \omega_i J_i$$

- Two complex-conjugate sectors with same $Re[S(Q,J)]$.

How...?

leading entropy $\propto N^2$

$$(\text{degeneracy with } \pm \text{ sign}) = e^{S_0(Q,J)} + e^{S_0(Q,J)^*} \sim e^{\mathrm{Re}(S_0)} \cos[\mathrm{Im}(S_0)]$$

Mostly determine the sign oscillation, also w/ small subleading entropy $\propto \log[\# \cos(\# N^2)]$.

- Realizes the sign-oscillation of the macroscopic degeneracies in an index.

[Agarwal, Choi, J. Kim, SK, Nahmgoong] (2020)

Constraints

- Naively, these constraints look weird. (The case with $Im(\sigma/\tau) > 0$, $\sum_I \delta_I - \sigma - \tau = -1$)

$$\mathrm{Im}\left(\frac{\sigma-\delta_I}{\tau}\right) < 0\ ,\quad \mathrm{Im}\left(\frac{1+\delta_I}{\tau}\right) < 0\ ,\quad \mathrm{Im}\left(\frac{1-\tau+\delta_I}{\sigma}\right) < 0\ ,\quad \mathrm{Im}\left(\frac{\delta_I}{\sigma}\right) > 0$$

- Interpretation: "Stability conditions" of Euclidean gravity dual against D3-brane instantons.

[Aharony, Benini, Mamroud, Milan] (2021)

Wraps $S^3 \subset S^5$, AdS_5: $S^3 \leftarrow S^1$(wrapped)

$\downarrow$

S^2 (transverse, N/S pole)

$$S_{1I} = 2\pi N \frac{\delta_I}{\sigma}\ ,\quad S_{2I} = 2\pi N \frac{\delta_I}{\tau} \qquad Z \leftarrow e^{iS_{iI}}$$

- $Im(S_{iI}) > 0$: Otherwise, transseries ruined. (Saddles presumably unstable.)
- These are stability conditions from only a selection of instantons.
- Our inequalities are stronger, suggesting more stability conditions.

- "Lorentzian signature" black holes violating this condition?
- Perhaps, "thermodynamic instability"
- Checked that the small black holes with high spin $J_1 - J_2$ can increase its entropy by "emitting" graviton hairs. [Choi, Jeong, SK] (2021)

Concluding remarks

- Due to the lack of time, omitted some interesting findings → Please see the paper.

- Future directions

- Connection between "excluding branch-point singularities" & "gravitational stability" …?

- In some regions, certain branch points approach arbitrarily close to our domain.
- Meaning, certain operators become very "light" in the background of black hole saddles.
- Comparing light QFT operators vs. light near-horizon BH modes…?

- Also found multi-cut saddles, sometimes with new continuous parameters. Gravity duals?

- More exact saddles?
- Exact AdS black holes in other dimensions, based on similar ideas?