East Asia Joint Symposium 21/11/22-21/11/26

M2-branes & Quantum Curves

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[M 2020 JHEP] [M-Yamada 2021 SIGMA]

DOF for Branes

[Klebanov-Tesytlin 1998]

- DOF for
  - D3-branes = N^2
  - M2-branes = $N^{3/2}$
  - M5-branes = N^3

M2-branes

- Worldvolume Theories

  = SUSY Chern-Simons Theories

  [Aharony-Bergman-Jafferis-Maldacena 2008]

- Free Energy = $N^{3/2}$

  [Drukker-Marino-Putrov 2009]

↓

Hints for Another Viewpoint of M2-branes
= Quantum Curves

Contents

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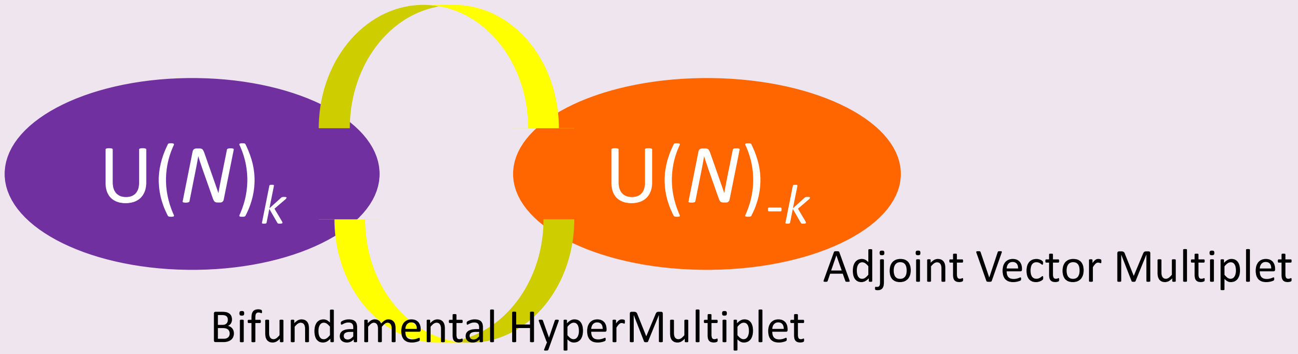
ABJM Theory

3D $\mathcal{N}=6$ Super Chern-Simons Theory



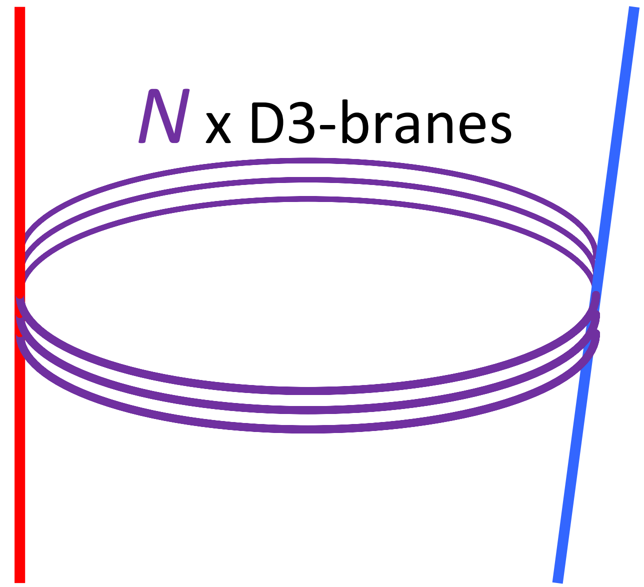
[Aharony-Bergman-Jafferis-Maldacena
Hosomichi-Lee-Lee-Lee-Park, ABJ 2008]

N M2-branes on C^4 / Z_k

Brane Configuration in IIB

From Large Supersymmetries

[Kitao-Ohta-Ohta 1998]



(IIB String Theory)

→ T-duality to IIA

→ Lift to M-Theory

ABJM Matrix Model

Partition Function

- Defined by Infinite-Dim Path Integral

  (Cancellations between Bosons & Fermions in SUSY Theories)

- Localized to Finite-Dim Matrix Integration

[Kapustin-Willett-Yaakov 2009]

ABJM : Gauge Group $U(N) \times U(N)$, Level k

$$Z_k(N) = \ldots$$

Expecting ...

Free Energy $\log Z_k(N)$ Reproduces DOF $N^{3/2}$ (?)

- 't Hooft Expansion

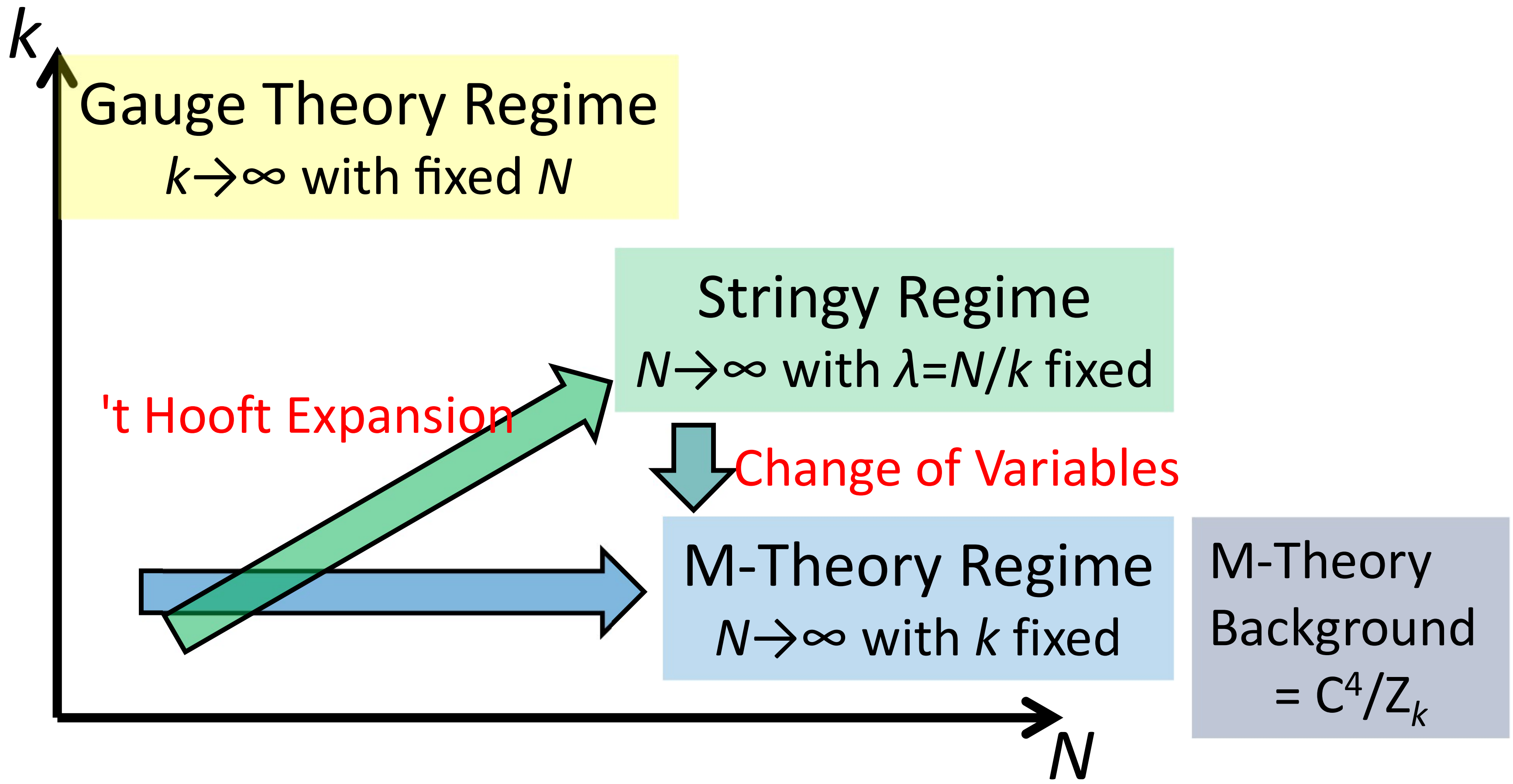
[Drukker-Marino-Putrov 2010]

$$\log Z_k(N) = (N/k)^{-1/2} N^2 \quad \text{(with 't Hooft Coupling } N/k \text{ Fixed)}$$

$$= k^{1/2} N^{3/2} \quad \text{(with M-Theory Background } k \text{ Fixed)}$$

$\rightarrow$ DOF $N^{3/2}$ + ...Corrections...

Dissatisfaction



DOF $N^{3/2}$ & Airy Function

- Summing Up Corrections

[Fuji-Hirano-M 2011]

$$\exp[N^{3/2} + \text{...Corrections...}] = \mathbf{Ai}(N) = \int \frac{d\mu}{2\pi i} e^{\frac{1}{3}\mu^3 - N\mu}$$

- Compared with Grand Potential $J_k(\mu)$

$$Z_k(N) = \int \frac{d\mu}{2\pi i} e^{J_k(\mu) - N\mu}$$

Move to Grand Canonical Ensemble

- Grand Partition Function $\Xi_k(z)$

$$\Xi_k(z) = \Sigma_{N=0}^{\infty} z^N Z_k(N)$$

(N : Particle Number, z : Dual Fugacity)

- Grand Potential

$$J_k(\mu) = \log \Xi_k(e^{\mu})$$

($\mu = \log z$: Chemical Potential)

Spectral Determinant

[Marino-Putrov 2011]

- Grand Partition Function = Fredholm Det

$$\Xi_k(z) = \mathrm{Det}(\, 1 + z\, H^{-1} \,)$$

with Spectral Operator

$$H = \mathcal{Q}\,\mathcal{P}$$

$$\mathcal{Q} = \boldsymbol{Q}^{1/2} + \boldsymbol{Q}^{-1/2},\ \mathcal{P} = \boldsymbol{P}^{1/2} + \boldsymbol{P}^{-1/2},\ \boldsymbol{Q}\,\boldsymbol{P} = e^{2\pi i k}\,\boldsymbol{P}\,\boldsymbol{Q}$$

$$(\ \boldsymbol{Q} = e^{q},\ \boldsymbol{P} = e^{p},\ [q,p] = i\,\hbar,\ \hbar = 2\pi k\ )$$

(After Similarity Transformations,

$$H = \boldsymbol{Q} + \boldsymbol{Q}^{-1} + \boldsymbol{P} + \boldsymbol{P}^{-1}$$

Reminiscent of Def. Eq. for $P^1 \times P^1$, But Quantized)

WKB Expansion

- Quantum-Mechanical Spectral Problem

  Spectral Operator H & Planck Const $\hbar = 2\pi k$

- Detecting M-Theory Regime (with fixed k)

From 't Hooft Expansion To WKB $\hbar$ Expansion

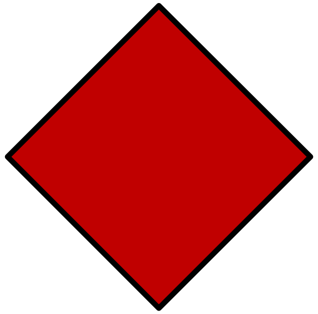
Phase Space Area

- Grand Potential from Phase Space Area

$$\partial_\mu [\, J_k(\mu)\,] = \partial_{\log z} [\, \log \Xi_k(z)\,]$$
$$= \partial_{\log z} [\, \log \mathrm{Det}(1 + z\, H^{-1})\,]$$
$$= \mathrm{Tr}\, (1 + H/z)^{-1} = \mathrm{Area}(H < z) / (2\pi\hbar)$$

- Classically ($\hbar = 2\pi k \to 0$)

$$\mathrm{Area}(H < z) = \mathrm{Area}\, (\, |q| + |p| = \mu\,) = \mu^2$$



Spectral Determinant

- As long as Polynomial Spectral Operator

$$H = H(\boldsymbol{Q}, \boldsymbol{P})$$

Always

$$\text{Area} = \mu^2$$

- After Integration

$$J_k(\mu) = \mu^3/3$$

Always Airy Functions

Proposal

- If we regard Airy Functions or $N^{3/2}$ as Characteristics of Multiple M2-branes,

Multiple M2-branes are described by Spectral Operators (not Matrix Models)

Non-Perturbative Effects

- Airy Function = just a starting point
- Full Exploration of Non-Perturbative Effects

…

[Hatsuda-M-Okuyama 2012, 2013, Hatsuda-Marino-M-Okuyama 2013]

Finally, Non-Perturbative Effects

- Redefinition of Chemical Potential μ

[Hatsuda-M-Okuyama 2013]

- Non-Perterbatively

  Free Energy of Topological Strings on Local P^1 x P^1

[Hatsuda-Marino-M-Okuyama 2013]

From Geometrical Viewpoint

- Redefinition of Chemical Potential μ

[Hatsuda-M-Okuyama 2013]

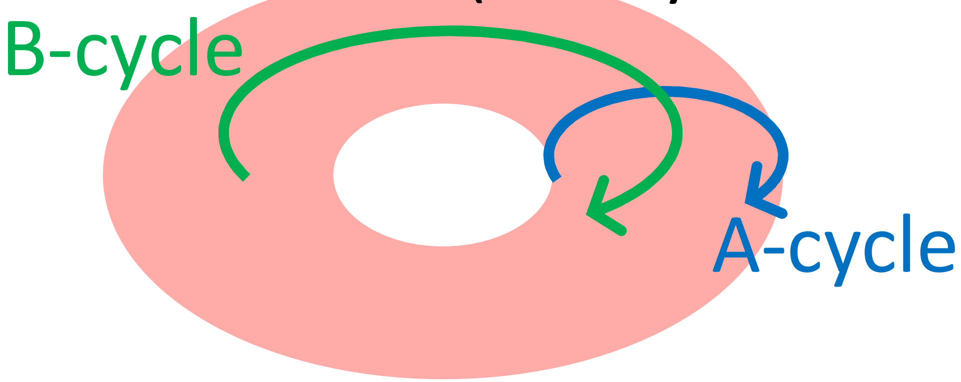
(Mirror Map, A-Period of P^1 x P^1)

- Non-Perterbatively

Free Energy of Topological Strings on Local P^1 x P^1

[Hatsuda-Marino-M-Okuyama 2013]

(Mostly B-Period of P^1 x P^1)



"Quantum Curves"

From Matrix Models To Curves

[Marino-Putrov 2011]

[Hatsuda-Marino-M-Okuyama 2013]

ABJM Matrix Model $\Xi_k(z)$

=

Spectral Det

Det ($1 + z H^{-1}$)

H = (Curve Eq of $\mathbf{P^1}$ **x** $\mathbf{P^1}$)

=

Free Energy of Top Strings

exp [$\Sigma N^d_{j_L,j_R} F^d_{j_L,j_R}(T)$]

$N^d_{j_L,j_R}$: BPS Index

on **Local** $\mathbf{P^1}$ **x** $\mathbf{P^1}$

d: degree, (j_L,j_R): spins

From Matrix Models To Curves

(Without Referring To Matrix Model)

"Spectral Theories/Topological Strings Correspondence"

[Grassi-Hatsuda-Marino 2014]

Spectral Det	=	Free Energy of Top Strings
$\mathrm{Det}\,(\,1 + z\,H^{-1}\,)$		$\exp\,[\,\Sigma\, N^d_{j_L,j_R}\, F^d_{j_L,j_R}(T)\,]$

H = (Curve Eq)

$Q = e^q$, $P = e^p$, $[q,p] = i\,2\pi k$

$N^d_{j_L,j_R}$: BPS Index on the Curve

Take-Home Message

Multiple M2-branes are described by Spectral Theories/Topological Strings Correspondence

Bonus:

Symmetries of Exceptional Weyl Group E_n

for Curves of Genus One (del Pezzo)

- However, Correpondence, Not Very Explicit

Contents

Again,

Spectral Theories/Topological Strings Correspondence,

for Curves of Genus One

(del Pezzo, Classified by Exceptional Weyl Group ... E6, E7, E8)

↑

Exceptional Weyl Group,

Crucial on both ST & TS sides

Contents

E8 Quantum Curve

[Moriyama 2020]

All Quantum Curves of Genus One (del Pezzo),
Especially Notoriously Complicated E8 Curve

$$\frac{H}{\alpha} = q^{-3}Q^3P^2$$
$$+q^{-2}Q^2P\left([2]_qF_1P + qF_3^2G_2^3H_5\right)$$
$$+q^{-1}Q\left(\left(([3]_q-1)F_2 + F_1^2\right)P^2 + q^{\frac{1}{2}}(F_3^2F_1G_2^3H_5 + F_2G_1 + F_3G_2H_1)P + qF_3^2G_2^3H_4\right)$$
$$+([2]_qF_2F_1 + ([4]_q-[2]_q)F_3)P^2 + \left(F_3^2F_2G_2^3H_5 + [3]_qF_3G_1 + F_2F_1G_1 + F_3F_1G_2H_1\right)P + \frac{E}{\alpha} + F_3^2G_2^3H_3P^{-1}$$
$$+qQ^{-1}P^{-2}\left(P + q^{-\frac{1}{2}}g_1\right)\left(P + q^{-\frac{1}{2}}g_2\right)$$
$$\times\left(\left(([3]_q-1)F_3F_1 + F_2^2\right)P^2 + q^{-\frac{1}{2}}(F_3^3G_2^3H_5 + F_3F_1G_1 + F_3F_2G_2H_1)P + q^{-1}F_3^2G_2^2H_2\right)$$
$$+q^2Q^{-2}P^{-3}\left(P + q^{-\frac{3}{2}}g_1\right)\left(P + q^{-\frac{1}{2}}g_1\right)\left(P + q^{-\frac{3}{2}}g_2\right)\left(P + q^{-\frac{1}{2}}g_2\right)\left([2]_qF_3F_2P + q^{-1}F_3^2G_2H_1\right)$$
$$+q^3Q^{-3}P^{-4}\left(P + q^{-\frac{5}{2}}g_1\right)\left(P + q^{-\frac{3}{2}}g_1\right)\left(P + q^{-\frac{1}{2}}g_1\right)\left(P + q^{-\frac{5}{2}}g_2\right)\left(P + q^{-\frac{3}{2}}g_2\right)\left(P + q^{-\frac{1}{2}}g_2\right)F_3^2$$

$$\Sigma_{n=0}^{4} z^nF_n = \Pi_{i=1}^{4}(1+zf_i), \quad \Sigma_{n=0}^{2} z^nG_n = \Pi_{i=1}^{2}(1+zg_i), \quad \Sigma_{n=0}^{6} z^nH_n = \Pi_{i=1}^{6}(1+zh_i)$$

Progress to date

- Quantum Affine D5 Weyl Group (q-Painleve eq)

[Hasegawa 2011]

- Classical Degenerate Curves (Seiberg-Witten curve)

"E6 = Highest Toric Curve"

[Benini-Benvenuti-Tachikawa 2009, Kim-Yagi 2014]

- Quantum D5, E6, E7 Curves (q-Heun eq)

[Takemura 2018]

- Quantum E8 Curve & Affine E8 Weyl Group

[M 2020, M-Yamada 2021]

Keys

- Basically, Weyl Ordering of Operators
- For Degenerate Curves,

Back&Forth Between "Triangle" & "Rectangle"

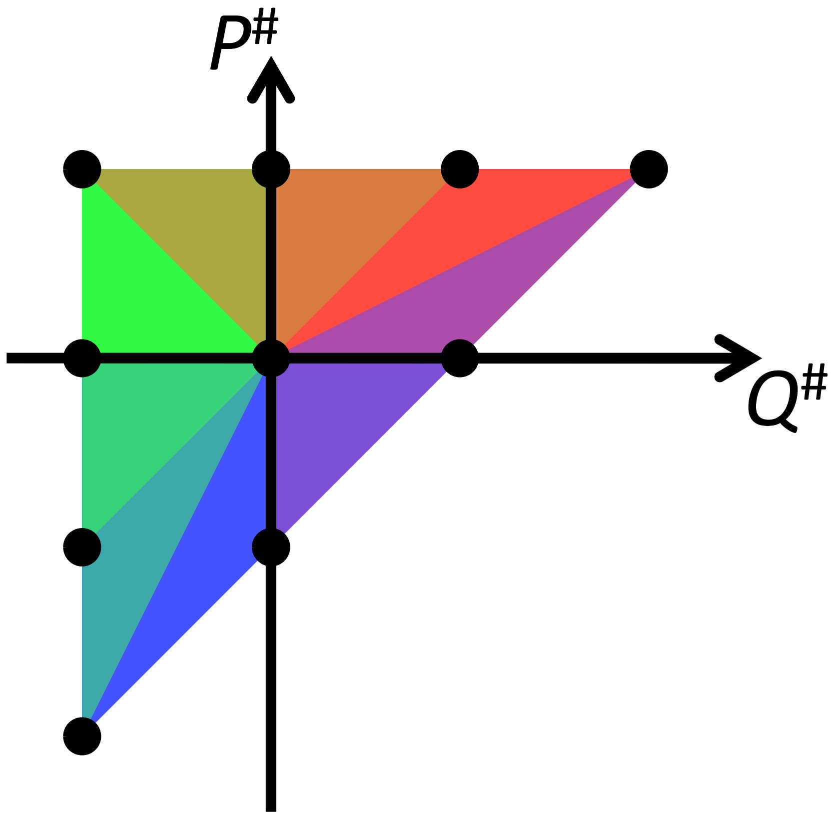
→ Step-by-Step Increasing Powers of q

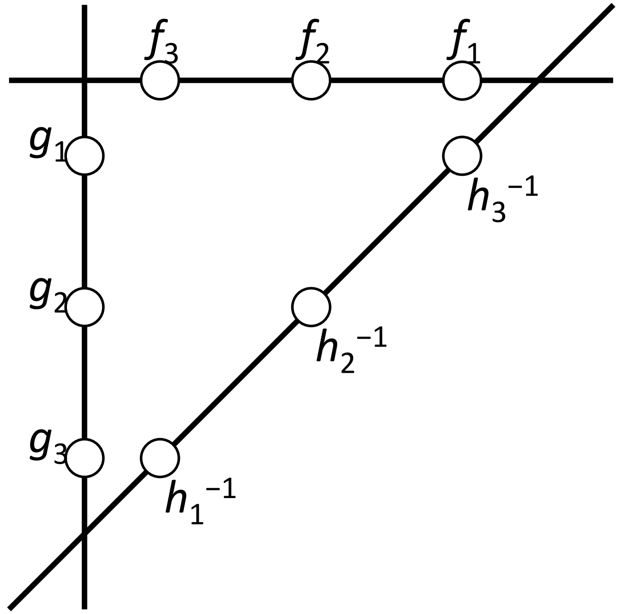
(DanDanBatake Structure)

（段々畑≈棚田=千枚田=梯田=계단식전=Rice Terraces）

Quantum E6 Curve

Triangular Realization

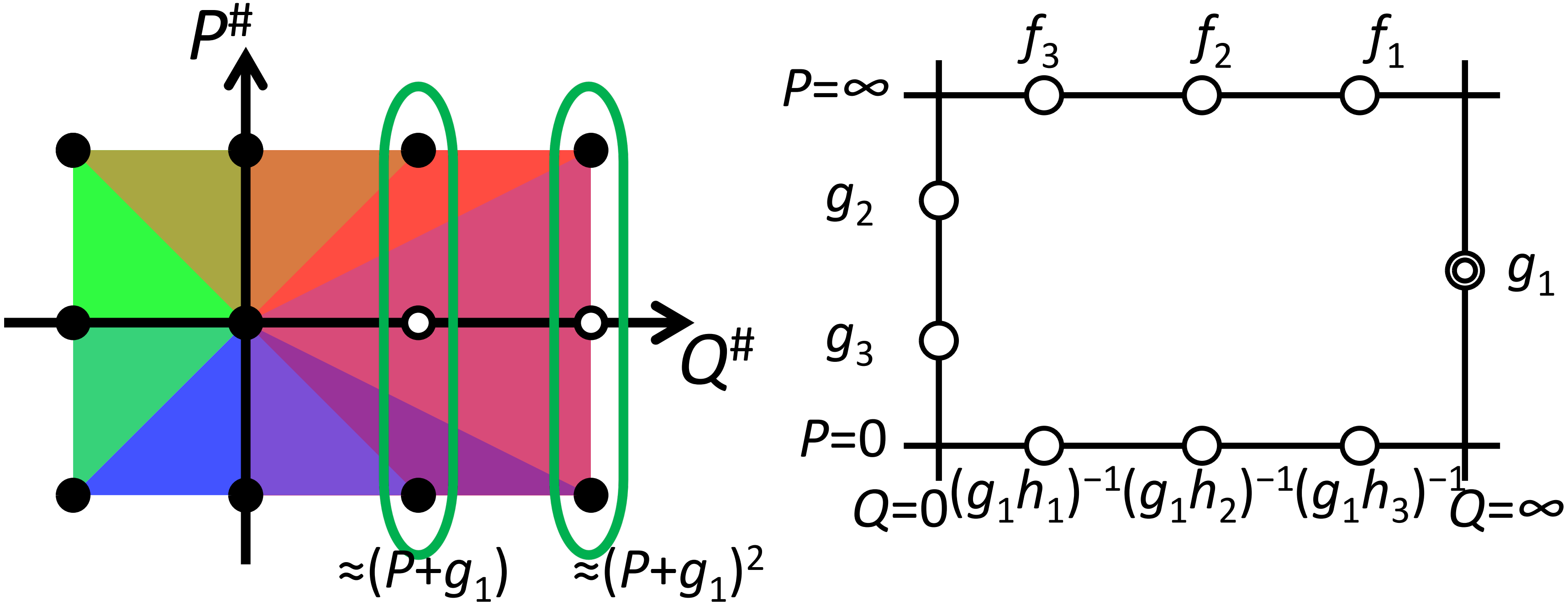




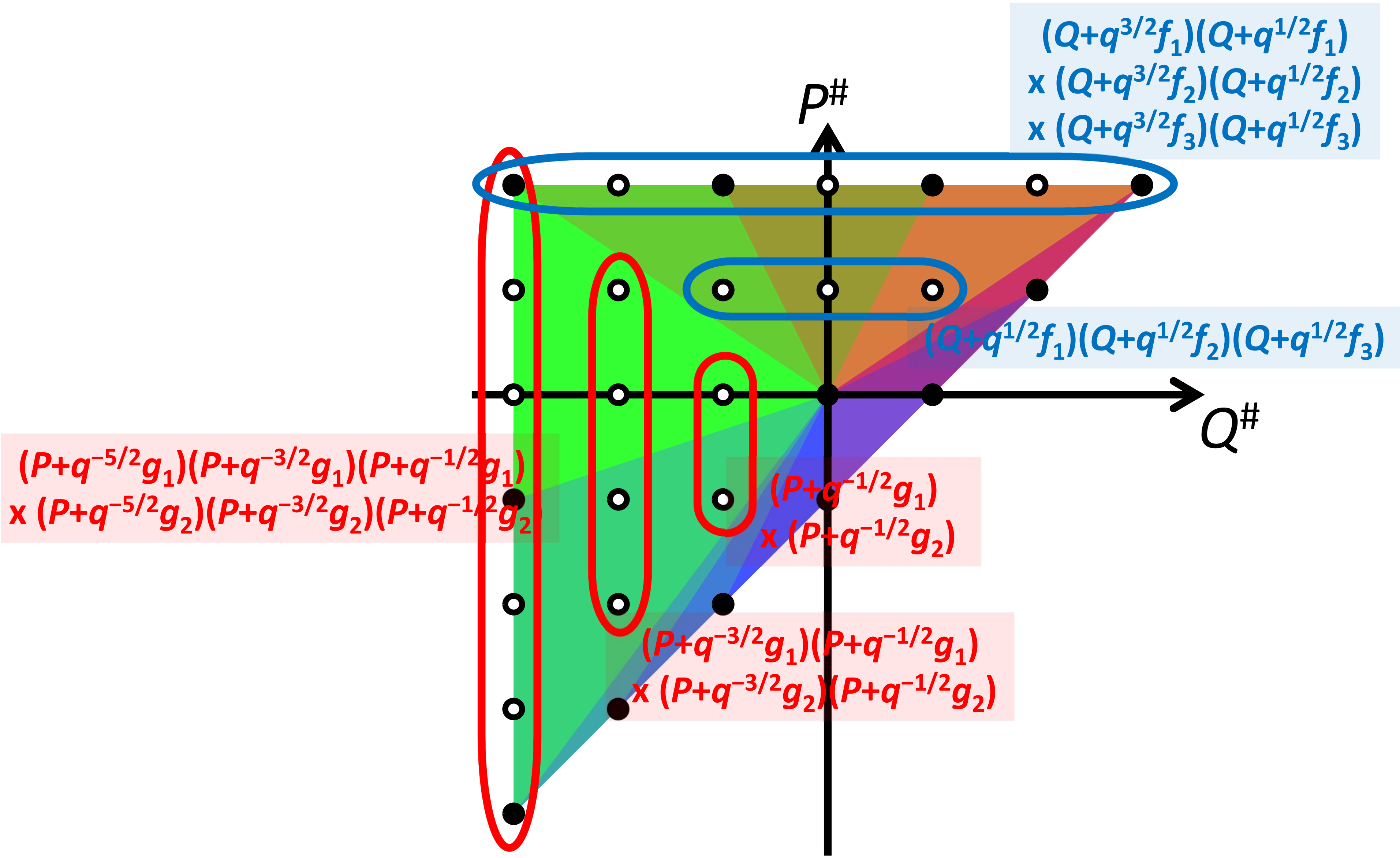
Quantum E6 Curve

Degenerate Rectangular Realization Via Similarity Transf.

$$Q' = P\left(P + q^{-\frac{1}{2}} g_1\right)^{-1} Q, \qquad P' = P$$



Quantum E8 Curve



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E8 Mirror Map

Requiring Weyl Group Invariance,

- Coefficients for Airy Functions
- BPS Indices (= Mostly B-Period)

studied extensively [Huang-Klemm-Poretschkin 2013]

- Also, Mirror Map (= A-Period)

Multi-Covering Structure, Same Reps for Both A & B,

From E8 to Lower Ranks via Group Decomposition

(Known for BPS Indices, Valid for Mirror Map as well [Sakai, private communication])

[Furukawa-M-Sugimoto 2019, M 2020]

Conjecture

- If Redefining Chemical Potential by

$$\mu = \mu_* + \sum_{l=1}^{\infty} (-1)^l A_l e^{-l\mu_*}, A_l = \sum_{n|l} \frac{(-1)^{n+1}}{n} \alpha_{\frac{l}{n}}(q^n), \alpha_d(q) = \sum_{j,R} \boldsymbol{m}_j^{d,R} (q^j + q^{-j}) \chi_R(\boldsymbol{q})$$

Indices for Mirror Map

Group Character

Then

$$J(\mu) = J_P(\mu_*) + J_W(\mu_*) + J_M(\mu_*)$$

- Perturbative Part (P)

Quadratic Term is Missing

$$J_P(\mu) = C\mu^3/3 + B\mu + A$$

$$C = \cdots, B = \cdots \text{ (2nd Casimir)}$$

Conjecture

- Worldsheet Instanton (W-Series)

$$J_W(\mu) = \sum_{m=1}^{\infty} d_m(k,\boldsymbol{b}) e^{-m\frac{\mu}{k}}, \quad d_m(k,\boldsymbol{b}) = (-1)^m \sum_{n|m} \frac{\delta_{m/n}(k/n, n\boldsymbol{b})}{n},$$

$$\delta_d(k,\boldsymbol{b}) = \frac{(-1)^{d-1}}{\left(2\sin\frac{\pi}{k}\right)^2} \sum_{j_L,j_R} \sum_{R} n_{j_L,j_R}^{d,R} \chi_R(e^{2\pi i \boldsymbol{b}}) \chi_{j_L}\left(e^{\frac{4\pi i}{k}}\right) \chi_{j_R}(1)$$

- Membrane Instanton (M-Series)

$$J_M(\mu) = \sum_{m=1}^{\infty} \left(b_l(k,\boldsymbol{b})\mu + c_l(k,\boldsymbol{b})\right) e^{-l\mu}, \quad b_l(k,\boldsymbol{b}) = \sum_{n|l} \frac{\beta_{l/n}(nk,\boldsymbol{b})}{n}, \ldots,$$

$$\beta_d(k,\boldsymbol{b}) = \frac{(-1)^d d}{4\pi \sin \pi k} \sum_{j_L,j_R} \sum_{R} n_{j_L,j_R}^{d,R} \chi_R(e^{2\pi i k \boldsymbol{b}}) \chi_{j_L}(e^{2\pi i k}) \chi_{j_R}(e^{2\pi i k})$$

BPS Indices

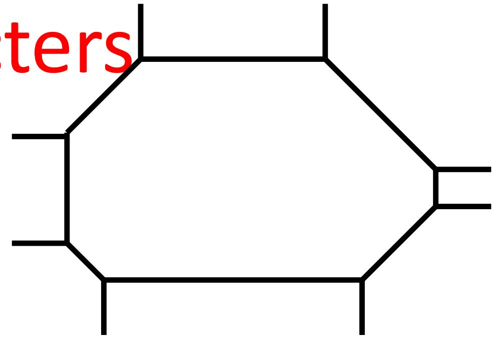
Group Character

su(2) Characters

Comment

- Kahler Parameters → Group Characters

  (Explicit Group Structure)



- Improving Previous Works

- Area of Dual Newton Diagram is 2nd Casimir Only After Shifting Chemical Potential Suitably

- Group Character Only After Identifying Overall Coeff. as Certain Combination of Parameters

[Mitev-Pomoni-Taki-Yagi 2014, Furukawa-M-Sugimoto 2019]

→ Overall Coefficient, Generally not Group-Inv

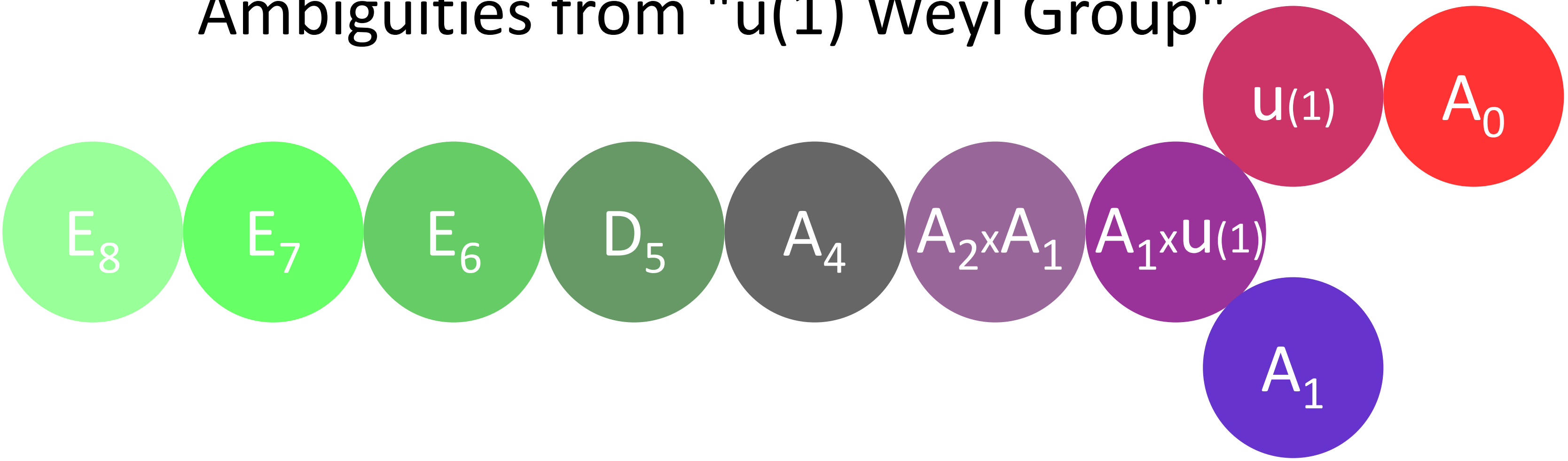
Comment

- Perturbative Quadratic Terms μ^2

[Gu-Klemm-Marino-Reuter 2015]

Del Pezzo Geometry of Lower Ranks

Ambiguities from "u(1) Weyl Group"



More Results

[M-Yamada 2021]

- Affine E8 Weyl Group including τ-Variables

  (Easier in Rectangular Realization)

- Characterizing Transf. Rule for τ-Variables

  "Fundamental Polynomials"

- DanDanBatake Structure

  = Non-Log Property of Difference Eq

- Bilinear Relations for τ-Variables

  → q-Painleve E8 Eq (See [Kajiwara-Noumi-Yamada 2015])

Contents

Summary

- Exploring M-Theory

From "Matrix Models"

To "Quantum Curves" !

"Not-Necessarily-Lagrangian Theories"

Summary

"Spectral Theories/Topological Strings Correspondence"

[Grassi-Hatsuda-Marino 2014]

Spectral Determinant

Det ($1 + z H^{-1}$)

=

Free Energy (Topological Strings)

$\exp [\Sigma N^{d}_{j_L,j_R} F^{d}_{j_L,j_R}(T)]$

Weyl Group Inv. for Both Sides

[M 2020, M-Yamada 2021]

Questions

Quantum Curves Enjoy only Finite Weyl Groups,
though q-Painleve Eqs Originate from Affine Weyl Group.
Where is Affine Weyl Group in M2-branes?

What are Symmetries of Quantum Curves
Beyond Genus One?

Higher Dimensional Phase Space for q-Painleve Eq.
What is Interpretation in M2-branes?

How about M5-branes?

Thank you for your attention.