Topological pseudo entropy

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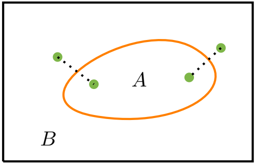
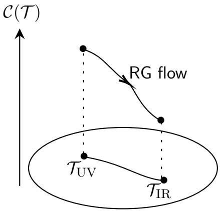
Introduction

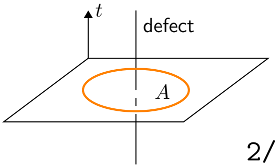
- Entanglement entropy in QFT:
  - can be an order parameter in various phase transitions [Calabrese-Cardy 04, Kitaev-Preskill 06, Levin-Wen 06, · · ·]
  - satisfies inequalities that constrain the dynamics of QFT ($\mathcal{C}$-theorems [Casini-Huerta 04, 12, Casini-Test´e-Torroba 17], ANEC [Faulkner-Leigh-Parrikar-Wang 16] etc)
  - probes non-local observables (boundary/interface/defect entropy [Nozaki-Takayanagi-Ugajin 12, Gaiotto 14, Estes-Jensen-O' Bannon-Tsatis-Wrase 14, Jensen-O'Bannon 15, Kobayashi-TN-Sato-Watanabe 18, Jensen-O' Bannon-Robinson-Rodgers 18, · · ·])



time slice



Space of QFTs



Goal of this talk

- We will examine pseudo entropy [Nakata-Takayanagi-Taki-Tamaoka-Wei 20], which measures quantum entanglement in a time-dependent system: $|\psi\rangle \xrightarrow[\text{time evolution}]{} |\varphi\rangle$
  - Entanglement entropy = von Neumann entropy of ρ_A :

$$\mathrm{Tr}_A\left[\rho_A\,\hat{\mathcal{O}}_A\right] = \langle\psi|\,\hat{\mathcal{O}}_A\,|\psi\rangle\ , \qquad \rho_A = \mathrm{Tr}_B\left[|\psi\rangle\langle\psi|\right]$$

  - Pseudo entropy = von Neumann entropy of $\tau_A^{\psi|\varphi}$:

$$\mathrm{Tr}_A\left[\tau_A^{\psi|\varphi}\,\hat{\mathcal{O}}_A\right] = \frac{\langle\varphi|\,\hat{\mathcal{O}}_A\,|\psi\rangle}{\langle\varphi|\psi\rangle}\ , \qquad \tau_A^{\psi|\varphi} \equiv \mathrm{Tr}_B\left[\frac{|\psi\rangle\langle\varphi|}{\langle\varphi|\psi\rangle}\right]$$

- We will address the following in simple setups:
  - How does it depend on topological data in TFT?
  - How different/similar is it to other measures?

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Entanglement entropy

Divide a system to A and $B = \bar{A}$: $\mathcal{H}_{\text{tot}} = \mathcal{H}_A \otimes \mathcal{H}_B$

Entanglement entropy

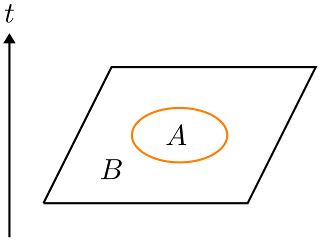
$$S_A = -\mathrm{Tr}_A\left[\rho_A \log \rho_A\right]$$

- The reduced density matrix:

$$\rho_A \equiv \mathrm{Tr}_B[\,\rho_{\text{tot}}\,]$$

- For a pure ground state $|\Psi\rangle$:

$$\rho_{\text{tot}} = |\Psi\rangle\langle\Psi|$$



Pseudo entropy

Pseudo Rényi entropy [Nakata-Takayanagi-Taki-Tamaoka-Wei 20]

$$S^{(n)}\left(\tau_A^{\psi|\varphi}\right) \equiv \frac{1}{1-n} \log \mathrm{Tr}_A \left[\left(\tau_A^{\psi|\varphi}\right)^n\right]$$

- $\tau_A^{\psi|\varphi}$: the reduced transition matrix for two states $|\psi\rangle, |\varphi\rangle$:

$$\tau_A^{\psi|\varphi} \equiv \mathrm{Tr}_B \left[\tau^{\psi|\varphi}\right] , \qquad \tau^{\psi|\varphi} \equiv \frac{|\psi\rangle \langle\varphi|}{\langle\varphi|\psi\rangle}$$

- $\tau_A^{\psi|\varphi}$ is not hermitian in general, so pseudo entropy can be complex:

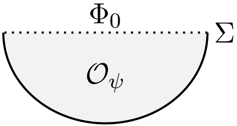
$$S^{(n)}\left(\tau_A^{\psi|\varphi}\right) = \left[S^{(n)}\left(\tau_A^{\psi|\varphi}\right)\right]^*$$

- Reduces to Rényi entropy when $\psi = \varphi$

Inner product $\langle\varphi|\psi\rangle$

- Let $\mathcal{O}_\psi$ be an operator corresponding to $|\psi\rangle$, then

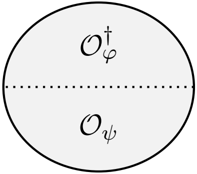
$$\langle\Phi_0|\psi\rangle = \int_{\Phi|_\Sigma=\Phi_0} \mathcal{D}\Phi\ \mathcal{O}_\psi\, e^{-I[\Phi]} =$$



- The inner product can be seen as a partition function with operator insertions:

$$Z\left[\mathcal{M}_1; \mathcal{O}_\psi, \mathcal{O}_\varphi^\dagger\right] \equiv \langle\varphi|\psi\rangle = \int \mathcal{D}\Phi_0\, \langle\varphi|\Phi_0\rangle\, \langle\Phi_0|\psi\rangle$$

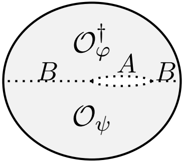
$$=$$



Transition matrix

- The unnormalized transition matrix:

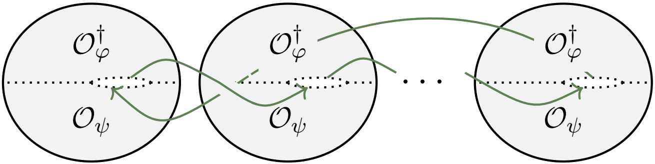
$$\tilde{\tau}_A^{\psi|\varphi} \equiv \mathrm{Tr}_B\left[|\psi\rangle\langle\varphi|\right] = \int \mathcal{D}[\Phi_0|_B]\,\langle\varphi|\Phi_0\rangle\,\langle\Phi_0|\psi\rangle =$$



- Gluing n copies of $\tilde{\tau}_A^{\psi|\varphi}$ to make the partition function on $\mathcal{M}_n$:

$$Z\left[\mathcal{M}_n; \mathcal{O}_\psi, \mathcal{O}_\varphi^\dagger\right] \equiv \mathrm{Tr}_A\left[\left(\tilde{\tau}_A^{\psi|\varphi}\right)^n\right]$$

$$=$$



Path integral representation of pseudo Rényi entropy

Pseudo Rényi entropy

$$S^{(n)}\left(\tau_A^{\psi|\varphi}\right) \equiv \frac{1}{1-n} \log \frac{Z\left[\mathcal{M}_n; \mathcal{O}_\psi, \mathcal{O}_\varphi^\dagger\right]}{Z\left[\mathcal{M}_1; \mathcal{O}_\psi, \mathcal{O}_\varphi^\dagger\right]^n}$$

- $Z\left[\mathcal{M}_n; \mathcal{O}_\psi, \mathcal{O}_\varphi^\dagger\right]$: the partition function on $\mathcal{M}_n$ with operators $\mathcal{O}_\psi, \mathcal{O}_\varphi$ inserted
- This representation makes manifest the property:

$$S^{(n)}\left(\tau_A^{\psi|\varphi}\right) = S^{(n)}\left(\tau_B^{\psi|\varphi}\right)$$

Chern-Simons theory and modular $\mathcal{S}$-matrix

- Chern-Simons theory on a 3d manifold $\mathcal{M}$ with gauge group $\mathrm{SU}(N)$ and level k:

$$I_{\mathsf{CS}}[A] = -\mathrm{i}\,\frac{k}{4\pi}\int_{\mathcal{M}} \mathrm{tr}\left[A \wedge \mathrm{d}A + \frac{2}{3}\,A \wedge A \wedge A\right]$$

- Wilson loops as a topological invariant observable:

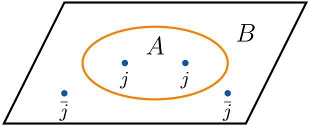
$$W_R[A] = \mathrm{tr}_R\,\mathcal{P}\exp\left(\int_C A\right)$$

- Partition functions are given by the modular $\mathcal{S}$-matrix [Witten 89]:

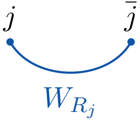
$$Z\left[\mathbb{S}^3\right] = \mathcal{S}_0{}^0$$
$$Z\left[\mathbb{S}^3; R_i\right] = \mathcal{S}_0{}^i$$

Pseudo entropy in Chern-Simons theory

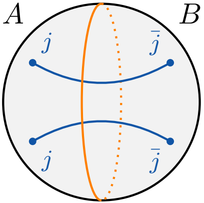
- In 3d $\mathrm{SU}(N)$ Chern-Simons theory with level k, consider a state with four quasi-particle excitations:

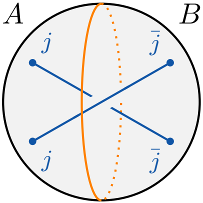


- A pair of excitations in conjugate representations are connected by a Wilson line in R_j representation

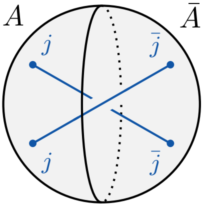


Example: two j's in A

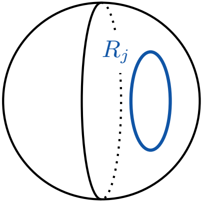
$|\psi\rangle =$ 

$|\varphi\rangle =$ 

- The resulting transfer matrix:

$\tilde{\tau}_A^{\psi|\varphi} =$ 

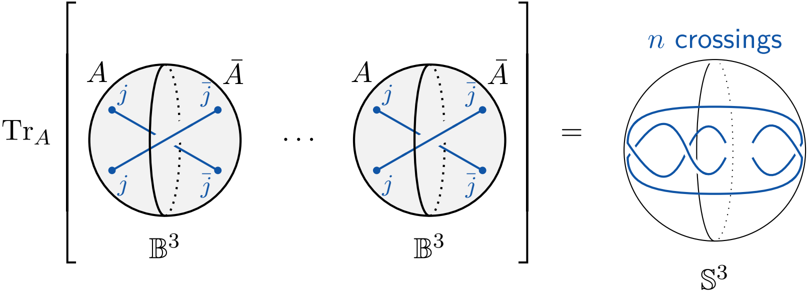
$\mathbb{B}^3$

$\Rightarrow \quad \langle\varphi|\psi\rangle =$ 

$\mathbb{S}^3$

Example: two j's in A

- Gluing n copies of $\tilde{\tau}_A^{\psi|\varphi}$ gives n Wilson loops:



- Pseudo entropy calculated by analytically continuing odd n:

$$S\left(\tau_A^{\psi|\varphi}\right) = \log \mathcal{S}_0{}^0 + \log \left[\frac{[N]}{[2]}\right] + \log \left[q^{\frac{1}{2}}\,[N+1] - q^{-\frac{1}{2}}\,[N-1]\right]$$

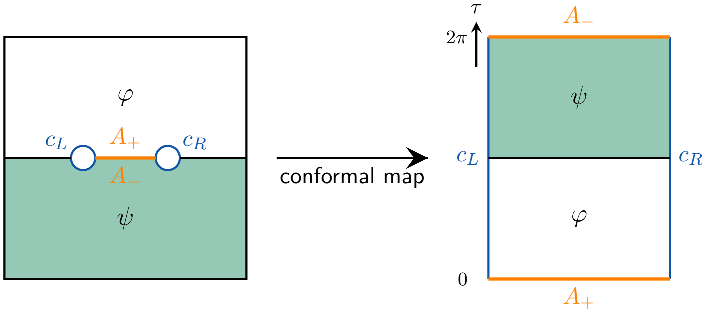
$$- \mathrm{i}\,\frac{\pi}{N+k}\,\frac{q^{\frac{1}{2}}\,[N+1] + q^{-\frac{1}{2}}\,[N-1]}{q^{\frac{1}{2}}\,[N+1] - q^{-\frac{1}{2}}\,[N-1]}$$

$(q = e^{2\pi \mathrm{i}/(N+k)})$

Pseudo entropy in CFT_2

For A a single interval in CFT_2, $^\exists$conformal map to a cylinder

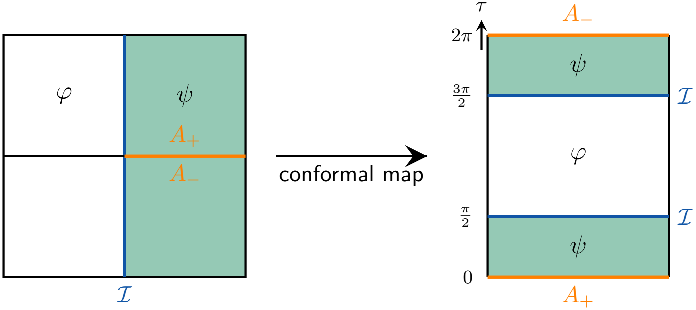
[Hislop-Longo 82, Casini-Huerta-Myers 11]



- $\tau \sim \tau + 2\pi$, no translation invariance along τ
- $c_{L,R}$: UV cutoffs around the entangling surface ∂A

Interface entropy

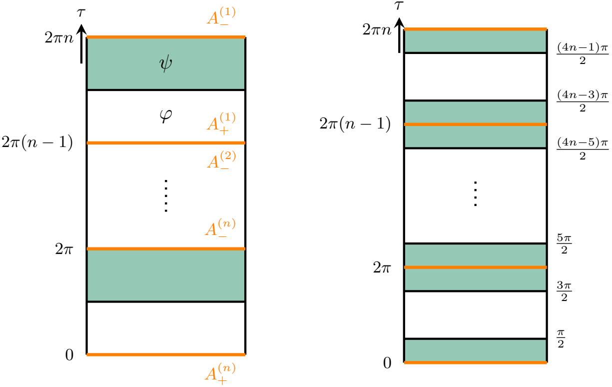
- A closely related measure is the entanglement entropy across a conformal interface $\mathcal{I}$ (interface entropy $S_A^{\mathcal{I}}$) [Sakai-Satoh 08, Gutperle-Miller 15, 17, Brehm-Brunner 15, Brehm-Brunner-Jaud-Schmidt-Colinet 15, Wen-Wang-Ryu 17, Chen-Hung-Li-Wan 18, Lou-Shen-Hung 19, Brehm 20]



- For states $|\psi\rangle, |\varphi\rangle$ glued along $\mathcal{I}$, the reduced density matrix $\rho_A^{\mathcal{I}}$ for $S_A^{\mathcal{I}}$ can be obtained by

$$\tau_A^{\psi|\varphi} \xrightarrow[\tau\to\tau+\pi/2]{} \rho_A^{\mathcal{I}}$$

Relation between pseudo entropy and interface entropy



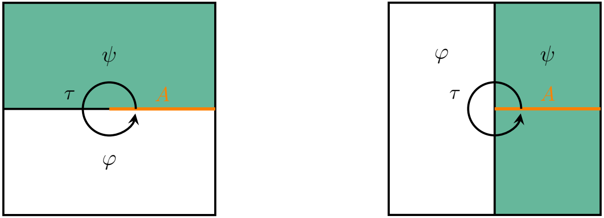
Pseudo entropy　　　　Interface entropy

Relation between pseudo and interface entropy in CFT_2

$$S\left(\tau^{\psi|\varphi}_{A:\,\text{interval}}\right) = S^{\mathcal{I}}_{A:\,\text{half-line}} \qquad \text{in } \mathrm{CFT}_2$$

Similar relation in QFT$_{d\geq 2}$

Taking the entangling surface to be a hyperplane in flat space:



Pseudo entropy   Interface entropy

Relation between pseudo and interface entropy in QFT$_{d\geq 2}$

$$S\left(\tau_A^{\psi|\varphi}\right) = S_A^{\mathcal{I}} \qquad \text{for} \quad \partial A : \{x^0 = x^1 = 0\}$$

Left-right entanglement entropy (LREE)

- In boundary CFT_2, LREE can be defined as the von Neumann entropy of the reduced density matrix for the left sector [Pando Zayas-Quiroz 14, Das-Datta 15]:

$$\rho_L^\psi \equiv \frac{1}{\langle\psi|\psi\rangle} \mathrm{Tr}_R \left[|\psi\rangle\langle\psi|\right]$$

- Any boundary state $|\psi\rangle$ satisfying the gluing condition

$$\left(L_n - \bar{L}_{-n}\right)|\psi\rangle = 0$$

can be expanded by the Ishibashi states [Ishibashi 89]:

$$|\psi\rangle = \sum_i \psi_i \, |i\rangle\!\rangle \,, \qquad \langle\!\langle i|j\rangle\!\rangle = \delta_{ij} \, \mathcal{S}_0{}^i$$

N.B. Boundary states are non-normalizable

Left-right pseudo entropy (LRPE)

- Pseudo entropy analogue of LREE can be defined with the transition matrix for two boundary states $|\psi\rangle, |\varphi\rangle$:

$$\tau_L^{\psi|\varphi} \equiv \frac{1}{\langle\varphi|\psi\rangle} \mathrm{Tr}_R \left[|\psi\rangle \langle\varphi| \right]$$

- Put a theory on a semi-infinite cylinder of circumference ℓ and regularize states by evolving along the imaginary time:

$$|\psi\rangle \to e^{-\epsilon H} |\psi\rangle \ , \qquad H = \frac{2\pi}{\ell} \left(L_0 + \bar{L}_0 - \frac{c}{12} \right)$$

- LRPE takes a complex value in general:

$$S\left(\tau_L^{\psi|\varphi}\right) = \frac{\pi c \ell}{24\,\epsilon} - \frac{\sum_i \mathcal{S}_i{}^0\, \psi_i\, \varphi_i^*\, \log(\psi_i\, \varphi_i^*)}{\sum_i \mathcal{S}_i{}^0\, \psi_i\, \varphi_i^*} + \log\left[\sum_i \mathcal{S}_i{}^0\, \psi_i\, \varphi_i^*\right]$$

Example

- Consider LRPE for the states:

$$|\psi\rangle = |a\rangle \ , \qquad |\varphi\rangle = |i\rangle\!\rangle$$

where $|a\rangle$ is a Cardy state [Cardy 89]

$$|a\rangle = \sum_i \frac{\mathcal{S}_a{}^i}{\sqrt{\mathcal{S}_0{}^i}} \, |i\rangle\!\rangle$$

- LRPE becomes

$$S\left(\tau_L^{\psi|\varphi}\right) = \frac{\pi c \ell}{24\,\epsilon} + \log\, \mathcal{S}_i{}^0$$

which is independent of the choice of the Cardy state $|a\rangle$ as long as it overlaps with the Ishibashi state $|i\rangle\!\rangle$

Summary and future directions

- A few analytic results for pseudo entropy obtained in topological and conformal field theories
- Non-trivial relation between pseudo and interface entropy derived
- Left-right pseudo entropy in BCFT introduced

- Supersymmetric generalization like supersymmetric Rényi entropy [TN-Yaakov 13]?
- Any application to constrain QFT dynamics?