Aspects of 5d Seiberg-Witten Theories on $\mathbb{S}^1$

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5D $N = 1$ supersymmetric field theory

Coulomb branch

- Symmetric breaking at generic points : $G \to \mathrm{U}(1)^r$
- Parametrized by real scalars ϕ^i $(i = 1, \cdots, r)$
- IMS* prepotential $\mathcal{F}_{\mathrm{IMS}}(\phi^i)$

Example: SU(2)-theory

- Prepotential:

$$\mathcal{F}_{\mathrm{IMS}}(\phi) = \mu_0 \phi^2 + \frac{4}{3}\phi^3 \quad (\mu_0 \equiv 8\pi^2/g_5^2) \tag{1}$$

- Coulomb branch:

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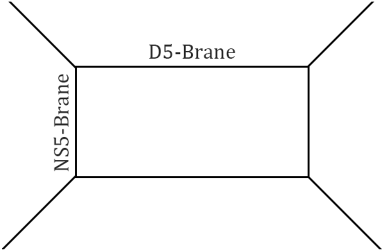
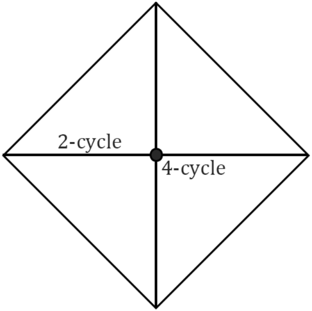
*Intriligator,Morrison,Seiberg 1997

String/M-theory realization

type IIB (p, q)-brane web

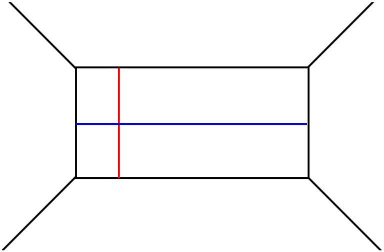


M-theory geometric engineering

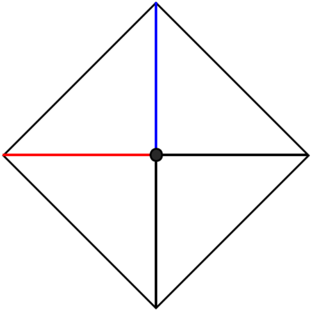


Electric BPS particles

(p,q)-string/string junction

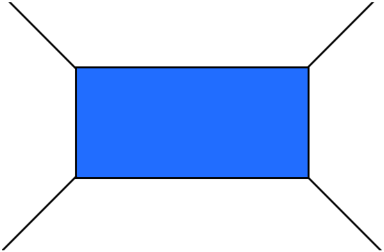


M2-brane wrapping 2-cycles

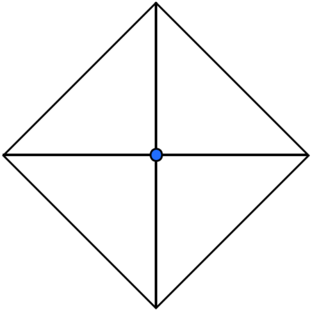


Magnetic BPS string

D3-brane wrapping common face



M5-brane wrapping 4-cycle

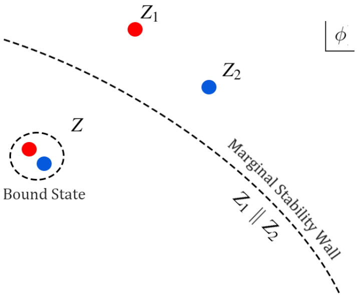


$\mathbb{S}^1$ Compactification

- $\phi \to \phi + iA_5$
- Prepotential receive instanton corrections
- magnetic string wrapping $\mathbb{S}^1 \to$ monopoles
- Wall-crossing is turning on

Wall-crossing

- BPS particles : $M = |Z(\phi)|$
- $Z = Z_1 + Z_2,\ M \leq M_1 + M_2$
- Marginal stability wall : $Z_1 \parallel Z_2$
- $M \leftrightarrow M_1 + M_2$ may happen when crossing the wall



Motivation

Complicate wall-crossing pattern in compactified theory:

- Various kinds of particles

$$Z \sim \overset{\text{Electric}}{n_e \cdot a} + \overset{\text{Magnetic}}{n_m \cdot ia_D} + \overset{\text{Instanton}}{n_I \cdot \mu_0} + \overset{\text{KK}}{n_k \cdot iR_5^{-1}}$$

- Whenever $Z_1 \parallel Z_2$, there could be a wall

On the other hand, there is no wall-crossing in 5D theory

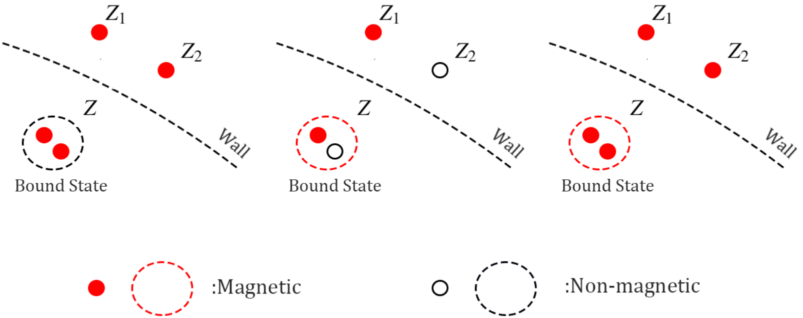
- No wall-crossing in particle sense
- Schwinger pairing between particles are zero

How does the wall-crossing turn itself off in $R_5 \to \infty$ limit?

Three kinds of wall-crossing

$$Z \rightarrow Z_1 + Z_2$$

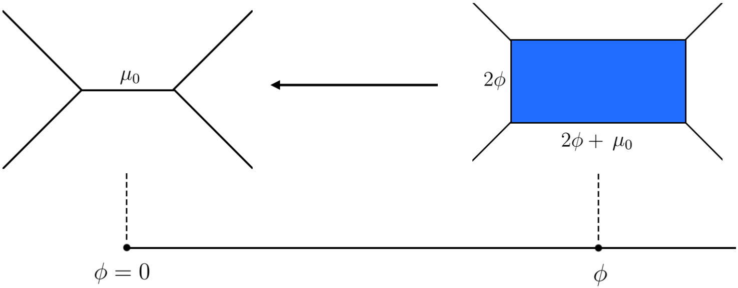
Z is non-magnetic Z_1 and Z_2 are magnetic	Z and Z_1 are magnetic Z_2 is non-magnetic	all three are magnetic



At least one of the two constituents Z_1, Z_2 must carry a magnetic charge

Case 1: Z is non-magnetic, Z_1 and Z_2 are magnetic

- $Z_{\text{magnetic}} \sim iR_5 T_{\text{magnetic}}$ should remain finite when $R_5 \to \infty$
- $T_{\text{magnetic}} \to 0$



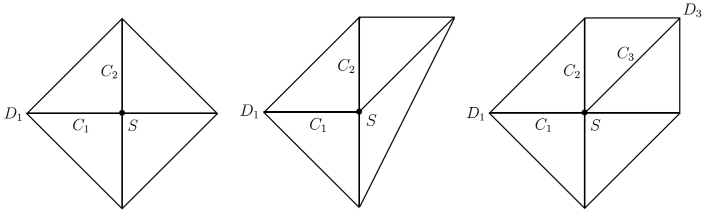
The marginal stability wall must collapse to the boundary of Coulomb branch, where magnetic strings become tensionless

Strategy

Exact prepotential :

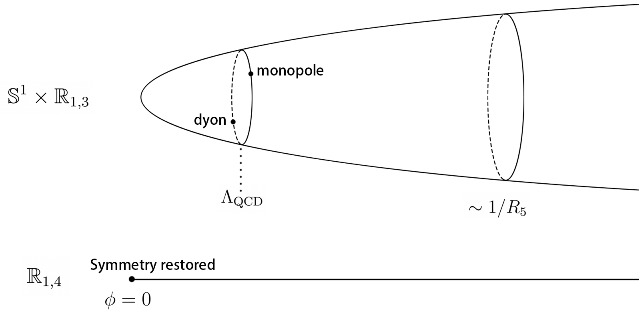
$$\frac{\partial^3 F_{\text{exact}}(t^i, v^a)}{\partial t^i \partial t^j \partial t^k} = S_i \cdot S_j \cdot S_k + \sum_{\eta \in H_2(X)} \frac{q^\eta}{1-q^\eta} N_\eta (S_i \cdot \eta)(S_j \cdot \eta)(S_k \cdot \eta)$$

- We use the exact prepotential to track the behaviour of Coloumb branch and marginal stability wall
- We analyse F0, F1 and dP_2 geometry separately



F0 example ($\mu_0 > 0$)

Coulomb branch:



- $g_{4,\rm eff}^{-2} \sim R_5 \cdot \mu_0 \gg 1$, $\Lambda_{\rm QCD} \gg \Lambda_{\rm UV} \sim 1/R_5$
- At $\sim \Lambda_{\rm QCD}$, there exists a wall:

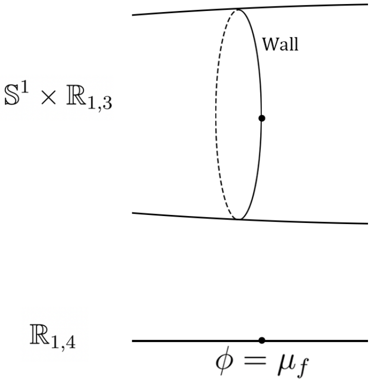
  $$(2,0)\ \text{W-Bosons} \to (0,1)\ \text{monopole} + (2,-1)\ \text{dyon}$$

  it shrinks to the $\phi = 0$ endpoint when $R_5 \to \infty$.
- For $Z_{\rm mag} \sim iR_5\phi(\phi + \mu_0)$ to be finite, $\phi \sim 1/R_5 \to 0$

Case 2: Z and Z_1 magnetic, Z_2 is non-magnetic

Coulomb branch (locally) :

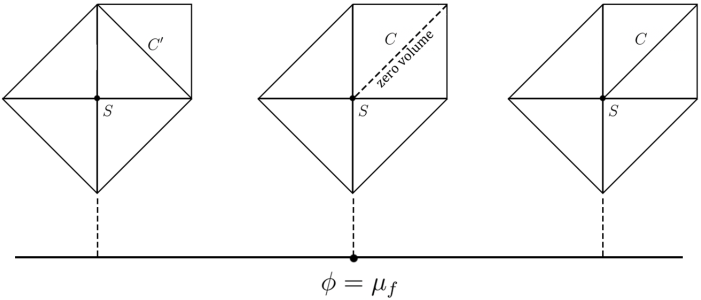
- The wall emanates form the point $Z_2 = 0$
- A flavor becomes massless
- $Z_2 \equiv Z_f = \pm(\phi + iA_5) + \mu_f$
- The wall extends in the circular direction



The wall collapse to a point in $R_5 \to \infty$ limit.

Discontinuity

dP$_2$ example (SU(2) with one flavor):



- Flop transition
- $\phi > \mu_f$: monopole carries the Jackiw-Rebbi zero mode[†]
- Discontinuity at $\phi = \mu_f$

[†] Jackiw,Rebbi 1976

"Wall-crossing" of magnetic string

- Magnetic string : M5-brane wrapping 4-cycle
- 6d (0,2) scft $\rightarrow$ 2d (0,4) scft[‡]
- The elliptic genus of the magnetic string changes:

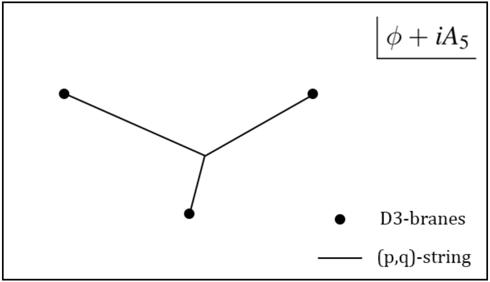
$$Z_{\text{ell},\phi>\mu_f} = Z_{\text{ell},\phi<\mu_f} \times \left(-\frac{\theta_{11}(q,y)}{\eta(q)}\right)$$

- The power of q is the KK-charge
- y is the fugacity of flavour charge
- A periodic chiral fermion generated by the Jackiw-Rebbi zero mode
- The same result is also obtained using KS wall-crossing formula

[‡]Maldacena,Strominger,Witten 1997

Case 3: all three are magnetic

- In 4d, such decays involving three kinds of dyons are known.
- in terms of 1/4-BPS states in $\mathcal{N} = 4$[§]



- Wall-crossing $\leftrightarrow$ breaking of the junction string
- It needs two independent adjoint scalars
- $R_5 \to \infty$, such states cannot exist.

[§] Lee, Yi 1998

Conclusion

- We consider the $\mathbb{S}^1$ compactification of 5D $\mathcal{N}=1$ SQFT
- Tracking the "wall-crossing" from 4d to 5d
- non-magnetic $\rightarrow$ magnetic + magnetic
  - Collapse to the boundary, turned off.
- magnetic $\rightarrow$ magnetic + non-magnetic
  - Discontinuity of the elliptic genus
- magnetic $\rightarrow$ magnetic + magnetic
  - Turned off