Wall-crossing of TBA equations and WKB periods for the higher order ODE

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Introduction

The ODE

$$\Big(-(-\hbar)^{r+1}\frac{d^{r+1}}{dx^{r+1}} + V(x) - E\Big)\psi(x) = 0$$

with polynomial potential $V(x)$ appears in many areas of physics:

- Schrödinger equation of 1D quantum mechanics when $r = 1$
- Minimal surface in the scattering amplitude/Wilson loop duality in $\mathcal{N} = 4$ SYM [Alday-Maldacena-Sever-Viera '10].
- Quantized Seiberg-Witten curve of (A_r, A_N) Argyres-Douglas theory in the NS limit of Ω background [Gaiotto '14, Ito-HS '17].
- Thermodynamic Bethe ansatz (TBA) equation in integrable system.

TBA and Resurgent Quantum Mechanics (r=1)

- WKB period: period integral along one-cycle on WKB curve $y^2 = V(x) - E$.
- Useful to write down the quantization condition.
- Featured by the asymptotic series and its discontinuity.

Voros' Riemann–Hilbert problem [Voros' 83]

Voros' Riemann–Hilbert problem: asymptotic series and the discontinuity $\rightarrow$ Exact WKB periods.

For the Schrödinger equation with polynomial potential

- The (Borel resummed) WKB periods are identified with the Y-functions, which satisfy the TBA equations [Ito-Marino-HS '18].
- TBA equations + exact quantization condition $\rightarrow$ exact (Voros) spectrum of the Schrödinger equation [Ito-Marino-HS, Gabai-Yin' 21].

(A_2, A_N)-type ODE

$$\left(\epsilon^3 \frac{d^3}{dx^3} + p(x)\right)\psi(x) = 0, \quad p(x) = u_0 x^{N+1} + u_1 x^N + \cdots + u_{N+1}$$

$$\Sigma_{\rm SW} \text{ of} (A_2, A_N) \text{ AD theory}: \; y^3 + p(z) = 0, \quad \lambda_{\rm SW} = y dz$$

WKB period

WKB ansatz: $\psi(x) = \exp\left(\frac{1}{\epsilon}\int^x P(x')dx'\right), \quad P(x) = \sum_{n=0}^{\infty} \epsilon^n p_n(x)$

Ricatti equation: $p(x) + P^3 + 3\epsilon P P' + \epsilon^2 P^{(2)} = 0$

WKB period: $\Pi_\gamma(\epsilon) = \int_\gamma P(x)dx = \sum_{n=0}^{\infty} \epsilon^n \Pi_\gamma^{(n)}, \quad \gamma \in H_1(\Sigma_{\mathrm{SW/WKB}})$

The quantum correction is expressed by the classical SW periods by acting the differential operator with respect to the moduli u_i:

$$\Pi_\gamma^{(n)} = \mathcal{O}_{\mathrm{PF}}^{(n)} \hat{\Pi}_{a\gamma}, \qquad \hat{\Pi}_{a\gamma} = \oint_\gamma (p(x))^{\frac{a}{3}} dx$$

One can solve the Ricatti equation recursively, and determine $\mathcal{O}_{\mathrm{PF}}^{(n)}$ order by order.

TBA/WKB correspondence

- $x \to \infty$: irregular singularity $\to$ Stokes phenomena.
- At infinity, the complex plane is divided into sectors, where the fastest decay solution is uniquely defined.

Y-function: cross ratios of the fastest decay solutions

(A_2, A_N) Y-system, Asymptotic behavior is govern by the classical period:

$$\log Y_{1,1} \sim \epsilon^{-1}\Pi^{(0)}_{\gamma_{1,1}} = m_{1,1}e^{\theta}, \quad \log Y_{1,2} \sim \left(\epsilon^{-1}\Pi^{(0)}_{\gamma_{1,2}}\right)^{[-1]} = m_{1,2}e^{\theta}, \cdots$$

for $\epsilon \to 0$. $\epsilon = e^{-\theta}, \phi_k = \arg(m_{1,k})$

Y-system + Asymptotics $\to$ N TBA equations:

$$\log Y_{1,k}(\theta - i\phi_k) = |m_{1,k}|e^{\theta} + K \star \overline{L}_{1,k} - K_{k,k-1} \star \overline{L}_{1,k-1} - K_{k,k+1} \star \overline{L}_{1,k+1},$$

$$2\pi K(\theta) = \frac{1}{\cosh(\theta + \frac{\pi i}{6})} + \frac{1}{\cosh(\theta - \frac{\pi i}{6})}, \quad K_{k_1,k_2}(\theta) = K(\theta - i(\phi_{k_1} - \phi_{k_2}))$$

Pole at $\phi_k - \phi_{k\pm 1} = \pi/3, 2\pi/3$.

Numeric test in minimal chamber $|\phi_k - \phi_{k\pm1}| < \pi/3$

$$\log Y_{1,1}(\theta) = \epsilon^{-1}\Pi_{\gamma_{1,1}}(\theta), \quad \log Y_{1,2}(\theta) = \left(\epsilon^{-1}\Pi_{\gamma_{1,2}}(\theta)\right)^{[-1]}, \cdots$$

Asymptotic series

ϵ expansion of WKB periods v.s. the $\epsilon = e^{-\theta}$-expansions of $\log Y_{1,k}(\theta)$:

$$\log Y_{1,k}(\theta) = m_{1,k}e^{\theta} + \sum_{n=1}^{\infty} m_{1,k}^{(n)} e^{-n\theta}$$

$$m_{1,k}^{(n)} \propto \int (\overline{L}_{a,k}(\theta)e^{n(\theta - i\phi_k)} - \overline{L}_{a,k-1}(\theta)e^{n(\theta-i\phi_{k-1})} - \overline{L}_{a,k+1}(\theta)e^{n(\theta-i\phi_{k+1})})d\theta$$

n	$\Pi_{\hat{\gamma}_{1,1}}^{(n)}$	$m_{1,1}^{(n-1)}$
0	$13.14579499i$	$13.14579499i$
2	$0.2172157436i$	$0.2172157436i$
6	$-1.519567945i$	$-1.519567945i$
8	$-20.48661777i$	$-20.48661776i$

Table: The quantum corrections for $p(x) = -x^3 + 7x + 6$. $m_{1,2}^{(n)} = -m_{1,1}^{(n)}$.

Also tested in many other examples.

Discontinuity

$$\log Y_{1,1}(\theta) = m_{1,1}e^{\theta} + \int_{-\infty}^{\infty} \mathrm{d}\theta' \Big(K(\theta-\theta'+i\phi_1)\overline{L}_{1,1}(\theta') - K(\theta-\theta'+i\phi_2)\overline{L}_{1,2}(\theta') \Big)$$

$$+\cdots, \quad \cdots$$

In the previous example: $\phi_1 = \frac{\pi}{2} = \phi_2$, poles are along the directions

$$\log Y_{1,1}(\theta): \qquad \theta = -\frac{5\pi}{6}, -\frac{\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}$$

$$\log Y_{1,2}(\theta - \frac{\pi i}{3}): \qquad \theta = -\frac{\pi}{2}, -\frac{5\pi}{6}, \frac{\pi}{6}, \frac{\pi}{2}$$

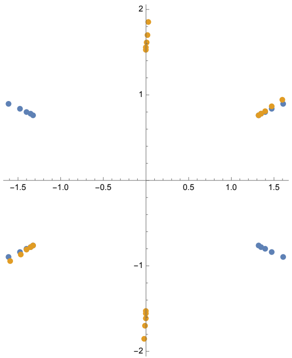

Figure: The singularity structure of the Borel transform of $\Pi_{\gamma_{1,1}}$ (blue) and $\Pi_{\gamma_{1,2}}$ (yellow) obtained by using the Borel-Padé technique applied to order ϵ^{160} terms.

TBA equations $\rightarrow$ Asymptotic series of WKB periods + discontinuity of the (Borel resummed) WKB periods $\rightarrow$ exact WKB periods.

Wall-crossing of (A_2, A_N) TBA equations

Wall-crossing of the TBA equations

Parameterize the zeros of $p(x)$ for (A_2, A_2) by
$x_0(t) = 3 - t, \quad x_1(t) = -1 + \sqrt{3}it, \quad x_2(t) = -2 + t - \sqrt{3}it, \quad 0 \le t \le 1.$

In the path $0 \le t \le 1$, we thus find two walls associated to $\phi_2 - \phi_1$:

$$t = 0.162117\ldots, \quad \phi_2 - \phi_1 = \frac{\pi}{3}, \quad \mathrm{Im}\left(\frac{\Pi^{(0)}_{\gamma_{1,2}}}{\Pi^{(0)}_{\gamma_{1,1}}}\right) = 0,$$

$$t = 0.397459\ldots, \quad \phi_2 - \phi_1 = \frac{2\pi}{3}, \quad \mathrm{Im}\left(\frac{\Pi^{(0)}_{\gamma_{2,2}}}{\Pi^{(0)}_{\gamma_{1,1}}}\right) = 0.$$

- Marginal stability walls locate at the pole in the kernel of TBA
- Crossing the wall, one needs to pick up the contribution of the pole and modify the TBA equations, i.e. wall-crossing of the TBA.

The 1st wall-crossing: $\phi_2 - \phi_1$ cross $\pi/3$

$$\log Y_{1,1}(\theta - i\phi_1) = |m_{1,1}|e^{\theta} + K \star \overline{L}_{1,1} - K_{1,2} \star \overline{L}_{1,2} - L_{1,2}\left(\theta - \frac{\pi i}{3} - i\phi_1\right),$$
$$\log Y_{1,2}(\theta - i\phi_2) = |m_{1,2}|e^{\theta} - K_{2,1} \star \overline{L}_{1,1} + K \star \overline{L}_{1,2} - L_{1,1}\left(\theta + \frac{\pi i}{3} - i\phi_2\right).$$

$$\log Y^{(1)}_{1,1}(\theta - i\phi_1) = |m_{1,1}|e^{\theta} + K \star \overline{L}^{(1)}_{1,1} - K_{1,2} \star \overline{L}^{(1)}_{1,2} + K^{-}_{1,12} \star \overline{L}^{(1)}_{12},$$
$$\log Y^{(1)}_{1,2}(\theta - i\phi_2) = |m_{1,2}|e^{\theta} + K \star \overline{L}^{(1)}_{1,2} - K_{2,1} \star \overline{L}^{(1)}_{1,1} - K^{-}_{2,12} \star \overline{L}^{(1)}_{12},$$
$$\log Y^{(1)}_{12}(\theta - i\phi_{12}) = |m_{12}|e^{\theta} + K \star \overline{L}^{(1)}_{12} + K^{+}_{12,1} \star \overline{L}^{(1)}_{1,1} - K^{+}_{12,2} \star \overline{L}^{(1)}_{1,2}.$$

$$\log Y^{(1)}_{1,1}(\theta) = e^{\theta}\Pi_{\gamma_{1,1}}, \quad \log Y^{(1)}_{1,2}(\theta) = e^{\frac{\pi i}{3}} e^{\theta}\Pi_{\gamma_{1,2}}(\theta + \tfrac{\pi i}{3})$$
$$\log Y^{(1)}_{12}(\theta) = e^{\theta}\Pi_{\gamma_{1,1}+\gamma_{1,2}}(\theta),$$ Test numerically!

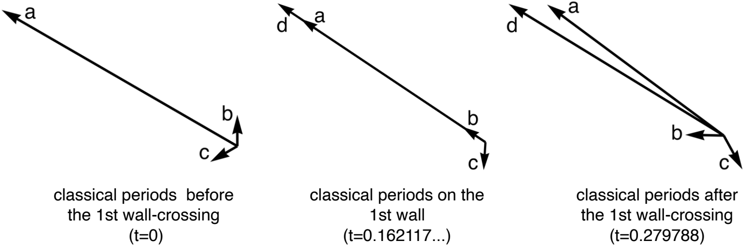

Figure: a,b,c,d and e: $\Pi^{(0)}_{\gamma_{1,1}}, \Pi^{(0)}_{\gamma_{1,2}}, \Pi^{(0)}_{\gamma_{2,2}}$ and $\Pi^{(0)}_{\gamma_{1,1}+\gamma_{1,2}}$.

Monomial point of (A_2, A_2): $p(x) = x^3 - 8$

The second wall-crossing occurs when $\phi_2 - \phi_1$ crosses $2\pi/3$. We introduce $Y^{(2)}_{\widetilde{12}}(\theta)$ whose mass is $m_{\widetilde{12}} = m_{1,1} + e^{-\frac{2\pi i}{3}} m_{1,2}$. Four-TBA equations

Symmetry of the classical periods/masses at $p(x) = x^3 - 8$

$|m_{1,1}| = |m_{1,2}| = |m_{12}|, \quad |m_{\widetilde{12}}| = \sqrt{3}|m_{1,1}|$
$\phi_2 - \phi_1 = \pi, \quad \phi_{12} - \phi_1 = \frac{\pi}{3}, \quad \phi_{\widetilde{12}} - \phi_1 = \frac{\pi}{6}$

We find $Y^{(2)}_{1,1}(\theta - i\phi_1) = Y^{(2)}_{1,2}(\theta - i\phi_2) = Y^{(2)}_{12}(\theta - i\phi_3)$

$$\begin{aligned}
\log Y^{(2)}_{1,1}(\theta - i\phi_1) &= |m_{1,1}|e^{\theta} + 3K(\theta-\theta') \star \overline{L}^{(2)}_{1,1} \\
&\quad + K(\theta-\theta'+\tfrac{\pi i}{6}) \star \overline{L}^{(2)}_{\widetilde{12}} + K(\theta-\theta'-\tfrac{\pi i}{6}) \star \overline{L}^{(2)}_{\widetilde{12}} \\
\log Y^{(2)}_{\widetilde{12}}(\theta - i\phi_1 - \tfrac{\pi i}{6}) &= \sqrt{3}|m_{1,1}|e^{\theta} + 3K(\theta-\theta') \star \overline{L}^{(2)}_{\widetilde{12}} \\
&\quad + 3K(\theta-\theta'+\tfrac{\pi i}{6}) \star \overline{L}^{(2)}_{1,1} + 3K(\theta-\theta'-\tfrac{\pi i}{6}) \star \overline{L}^{(2)}_{1,1}.
\end{aligned}$$

D_4-type TBA [Klassen Melzer, Bradenet al., Zamolodchikov]

Monomial point of (A_2, A_3): $p(x) = x^4 - 81$

- (A_2, A_3): 9 wall-crossing $\rightarrow$ 12 TBA equations in maximal chamber.
- The monomial point is symmetric as well.
- 12 TBA equations reduce to 4 TBA equations.
- E_6 TBA equations [Klassen Melzer, Bradenet al., Zamolodchikov].

Interpretation from Argyres-Douglas theory

- The (A_2, A_2) AD theory and D_4 AD theory have the common AD point.
- The (A_2, A_3) AD theory and E_6 AD theory have the common AD point $y^3 + x^4 = 0$ [Cecotti et.al '10, Xie '12].
- They can be regarded as the equivalent theory.
- The ODE with the monomial potential can be interpreted as the quantum SW curve of $D4/E_6$ AD theory.

Conclusions

- We have generalized the TBA/WKB correspondence to the 3rd equaton with general polynomial potential.
- We found $\log Y \sim \Pi$, which allows us to compute the WKB periods (quantum periods) exactly.
- At the monomial point, (A_2, A_2) TBA $\xrightarrow{2\, wall-ccrossing}$ D_4 TBA,
  (A_2, A_3) TBA $\xrightarrow{9\, wall-ccrossing}$ E_6 TBA
- The duality at quantum level: $(A_2, A_2) \sim D_4, \quad (A_2, A_3) \sim E_6$.
- Different integrable systems are unified by the same ODE but with different moduli parameters.

Thanks for your attention !