Free energy and defect C-theorem in free theory

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based on
arXiv:2101.02399 with T. Nishioka & arXiv:2102.11468
also arXiv:1810.06995 with N. Kobayashi, T. Nishioka, K. Watanabe

Introduction: Motivation

- Defect & Boundary appear in various areas and play important roles.
  - QFT: Wilson loop, 't Hooft loop
  - String Theory: D-brane, M-brane
- Due to a recent development of quantum information & condensed matter, importance of defects is again recognized.

  $\rightarrow$ I want to understand general properties of defects & boundary!!

  In particular, I focus on RG flow and C-theorem.
- It is difficult to treat defects & boundaries since they are non-local.

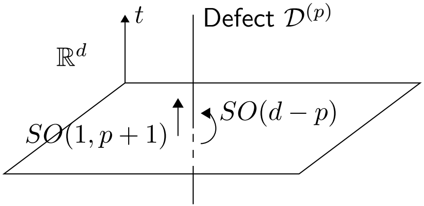
  $\rightarrow$ I focus on *conformal defect* which keeps enough symmetry.

Introduction: What are DCFT and BCFT?

- Euclidean d-dim Conformal Field Theory (CFT) has $SO(1, d+1)$ symmetry.

  When p-dim Defect exists, the allowed maximal symmetry is $SO(1, p+1) \times SO(d-p)$.

  $\rightarrow$ The theory is called Defect CFT (DCFT), and the Defect is called Conformal Defect.



- In particular, we can construct theories with boundary when $p = d-1$.

  $\rightarrow$ This is called Boundary CFT (BCFT).

Introduction: C-theorem (without Boundary nor Defect)

- C-theorem: There exists a monotonically decreasing function along the renormalization group,

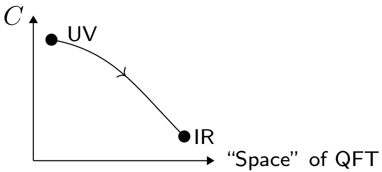
$$I_{\text{CFT}} + \lambda \int \mathrm{d}^d x \sqrt{g}\, \mathcal{O}(x)\,, \qquad \Delta_{\mathcal{O}} < d\,.$$

The C-function counts an effective degree of freedom of the theory.

Weak C-theorem:

The value at UV is always larger than IR.

$$C_{\text{UV}} \geq C_{\text{IR}}\,.$$



- Conjecture (proved in $d = 2, 3, 4$):

Free energy on sphere, $\tilde{F} \equiv \sin(\pi d/2)\, \log\, Z[\mathbb{S}^d]$, satisfies the weak C-theorem.

[Zamolodchikov '86, Cardy '88, Komargodski-Schwimmer '11, Myers-Sinha '10, Jefferis et al. '11, Klebanov-Pufu-Safdi '11,...]

Motivation of our work

- We proposed defect free energy is a C-function. [Kobayashi-Nishioka-YS-Watanabe '18]

$$\tilde{D} \equiv \sin\left(\frac{\pi p}{2}\right) \left(F_{\mathsf{DCFT}}[\mathbb{S}^d] - F_{\mathsf{CFT}}[\mathbb{S}^d]\right) .$$

under the RG flow localising on the defect,

$$I = I_{\mathrm{DCFT}} + \hat{\lambda} \int \mathrm{d}^p \hat{x} \, \sqrt{\hat{g}} \, \hat{\mathcal{O}}(\hat{x}) \, .$$

Checked our conjecture using various setups in field theory & holography.

- I want to check the proposal in simple models, free theories. [Nishioka-YS '21, YS '21]

- Problems: How to impose a conformal boundary condition on $\mathbb{S}^d$?

  i.e. How to construct DCFT in the free scalar field?

$\rightarrow$ Conformal map to $\mathbb{H}^{p+1} \times \mathbb{S}^{q-1}$ (p: dim of defect, $q = d - p$)

We can use methods familiar in AdS/CFT.

Conformally coupled scalar field on $\mathbb{H}^{p+1} \times \mathbb{S}^{q-1}$

- Scalar field on flat space: defect sits at $y_i = 0$

$$\mathrm{d}s^2 = \mathrm{d}x_a^2 + \mathrm{d}y_i^2 \,, \qquad (a = 1, \cdots, p, i = p+1, \cdots, d)$$

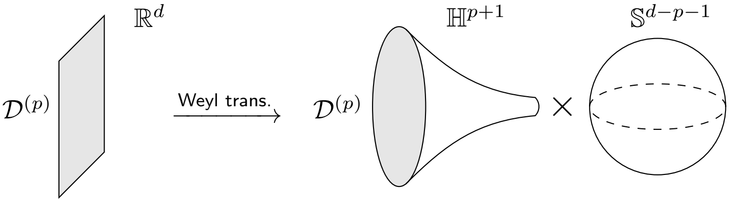
- Conformal map to $\mathbb{H}^{p+1} \times \mathbb{S}^{q-1}$ ($q = d - p$: co-dimension of the defect):

$$\mathrm{d}s^2 = \mathrm{d}x_a^2 + \mathrm{d}z^2 + z^2 \mathrm{d}s^2_{\mathbb{S}^{q-1}} = z^2 \left(\frac{\mathrm{d}x_a^2 + \mathrm{d}z^2}{z^2} + \mathrm{d}s^2_{\mathbb{S}^{q-1}} \right)$$

Now, the defect sits at $z = 0$ (i.e. boundary of $\mathbb{H}^{p+1}$).

We can regard theories on $\mathbb{H}^{p+1} \times \mathbb{S}^{q-1}$ as DCFT$_d$ with p-dim defect!!

(if we impose a suitable boundary condition.)



Boundary condition

- To preserve conformal symmetry, we need to impose a boundary condition at $z = 0$.
- Action on $\mathbb{H}^{p+1} \times \mathbb{S}^{q-1}$

$$I = \frac{1}{2} \int \mathrm{d}^d x \sqrt{g} \left[(\partial_\mu \phi)^2 + \xi \mathcal{R} \phi^2 \right] , \quad \xi = \frac{d-2}{4(d-1)} , \quad \mathcal{R} = \frac{(q-1)(q-2) - p(p+1)}{R^2}$$

- Decomposition: $\phi(z, x, \theta) = \sum_\ell \phi_{\mathbb{H}^{p+1}}(z, x) Y_{\ell, \mathbb{S}^{q-1}}(\theta)$

with spherical harmonics $Y_{\ell,\mathbb{S}^{q-1}}$: $-\nabla^2_{\mathbb{S}^{q-1}} Y_{\ell,\mathbb{S}^{q-1}}(\theta) = \frac{\ell(\ell+q-2)}{R^2} Y_{\ell,\mathbb{S}^{q-1}}(\theta)$

- Asymptotic behaviour of $\phi_{\mathbb{H}^{p+1}}$:

$$\Longrightarrow \quad \phi_{\mathbb{H}^{p+1}} \sim c_+^\ell z^{\Delta_+^\ell} + c_-^\ell z^{\Delta_-^\ell} , \quad \left[\Delta_\pm^\ell = \frac{p}{2} \pm \left| \ell + \frac{q-2}{2} \right| \quad (q \geq 2) , \quad \Delta_\pm = \frac{p}{2} \pm \frac{1}{2} \quad (q = 1) \right]$$

If we set $c_+^\ell = 0$ or $c_-^\ell = 0$, the theory preserves conformal symmetry.

- We want to consider unitary theories.
- Unitarity bound: $\Delta \geq \frac{p}{2} - 1 \quad (p \geq 2)\,,$ or $\Delta \geq 0 \quad (p < 2)$ $\quad \left(\Delta_{\pm}^{\ell} = \frac{p}{2} \pm \left|\ell + \frac{q-2}{2}\right|\right)$

  Δ_{+}^{ℓ} always satisfy unitarity bound. $\rightarrow$ Dirichlet b.c.

  Δ_{-}^{ℓ} satisfy unitarity bound if $\ell \leq 2 - \frac{q}{2}$.

  - $q = 2$:

    Nontrivial b. c. $\Delta_{\mathrm{N1}} = \begin{cases} \Delta_{+}^{\ell \neq 1} \\ \Delta_{-}^{\ell = 1} \end{cases}, \quad \Delta_{\mathrm{N2}} = \begin{cases} \Delta_{+}^{\ell \neq \pm 1} \\ \Delta_{-}^{\ell = \pm 1} \end{cases}$ $\rightarrow$ Neumann b.c..

  - $q = 3, 4$:

    Nontrivial b. c. $\Delta_{\mathrm{N}} = \begin{cases} \Delta_{+}^{\ell \neq 0} \\ \Delta_{-}^{\ell = 0} \end{cases}$ $\rightarrow$ Neumann b.c..

- Consistent with a classification by [Lauria-Liendo-van Rees-Zhao '20].

Defect C-theorem

- We want to check our proposed defect C-theorem in the free scalar.
- We assume that the difference of the C-function is not changed under the Weyl transformation,

$$\tilde{D}_{\mathrm{UV}}[\mathbb{S}^d] - \tilde{D}_{\mathrm{IR}}[\mathbb{S}^d] = \sin\left(\frac{\pi p}{2}\right)\left(F_{\mathrm{UV}}[\mathbb{H}^{p+1}\times\mathbb{S}^{q-1}] - F_{\mathrm{IR}}[\mathbb{H}^{p+1}\times\mathbb{S}^{q-1}]\right)$$

  UV $=$ Neumann b.c., IR $=$ Dirichlet b.c.

- We compute $F_{\mathrm{UV}}[\mathbb{H}^{p+1}\times\mathbb{S}^{q-1}]$ and $F_{\mathrm{IR}}[\mathbb{H}^{p+1}\times\mathbb{S}^{q-1}]$ explicitly.
- Double trace deformation triggers RG flows
  1. $q = 1, 3, 4$: $\Delta_{\mathrm{N}} \to \Delta_{\mathrm{D}}$
  2. $q = 2$: $\Delta_{\mathrm{N2}} \to \Delta_{\mathrm{N1}}$ or $\Delta_{\mathrm{N1}} \to \Delta_{\mathrm{D}}$

$q=1$, Free energy on $\mathbb{H}^d$

- Free energy of a conformally coupled scalar on $\mathbb{H}^d$:

$$F[\mathbb{H}^d]=\frac{1}{2}\operatorname{tr}\log\left[-\tilde{\Lambda}^{-2}\left(\nabla^2_{\mathbb{H}^d}+\frac{d(d-2)}{4R^2}\right)\right]=\frac{1}{2}\int_0^\infty \mathrm{d}\omega\,\mu^{(d)}(\omega)\log\left(\frac{\omega^2+\nu^2}{\tilde{\Lambda}^2R^2}\right)$$

1. $\tilde{\Lambda}$: UV cutoff scale introduced to make the integral dimensionless
2. $-\nabla^2_{\mathbb{H}^d}\phi_\omega=\left(\frac{\omega^2}{R^2}+\left(\frac{d-1}{2R}\right)^2\right)\phi_\omega$
3. $\Delta(\Delta-d+1)=-\frac{d(d-2)}{4},\quad \nu=\Delta-\frac{d-1}{2}$
4. Plancherel measure $\mu^{(d)}(\omega)$ on $\mathbb{H}^d$ of unit radius:

$$\mu^{(d)}(\omega)=\frac{1}{\Gamma(d)}\begin{cases}(-1)^{\frac{d-1}{2}}\,\frac{2}{\pi}\,\log(R/\epsilon)\prod_{j=0}^{\frac{d-3}{2}}(\omega^2+j^2) & d:\text{odd}\\ (-1)^{\frac{d}{2}}\,\omega\,\tanh(\pi\omega)\,\prod_{j=\frac{1}{2}}^{\frac{d-3}{2}}(\omega^2+j^2) & d:\text{even}\end{cases}$$

Comments on boundary condition

- Free energy of a conformally coupled scalar on $\mathbb{H}^d$:

$$F[\mathbb{H}^d] = \frac{1}{2}\int_0^\infty \mathrm{d}\omega\, \mu^{(d)}(\omega) \log\left(\frac{\omega^2+\nu^2}{\tilde{\Lambda}^2 R^2}\right) , \qquad \nu = \Delta - \frac{d-1}{2}$$

- The square integrability condition is implicitly imposed.

  $\rightarrow$ This expression is valid for Dirichlet b.c. $\Delta_+ = \frac{d}{2}$ (or $\nu = \frac{1}{2}$).

- Neumann b.c. $\Delta_- = \frac{d}{2} - 1$ (or $\nu = -\frac{1}{2}$) cannot be obtained from this expression directly.

  The integral expression of the free energy does not depend on the sign of ν.

  $\rightarrow$ We compute $F[\mathbb{H}^d]$ as a function of ν and analytically continue to negative ν.

- To compute the free energy, we use a zeta function regularisation.

Results of $\mathbb{H}^d$

- Odd d:

$$F_{\Delta_+}[\mathbb{H}^d] - F_{\Delta_-}[\mathbb{H}^d] = \log\left(\frac{\epsilon}{R}\right) \times \begin{cases} -\frac{1}{24} & d=3 \\ \frac{17}{11520} & d=5 \\ -\frac{367}{1935360} & d=7 \\ \frac{27859}{928972800} & d=9 \end{cases}$$

- Even d:

$$F_{\Delta_+}[\mathbb{H}^d] - F_{\Delta_-}[\mathbb{H}^d] = \begin{cases} \text{diverge (due to zero mode)} & d=2 \\ -\frac{\zeta(3)}{8\pi^2} & d=4 \\ \frac{\zeta(3)}{96\pi^2} + \frac{\zeta(5)}{32\pi^4} & d=6 \\ -\frac{\zeta(3)}{720\pi^2} - \frac{\zeta(5)}{192\pi^4} - \frac{\zeta(7)}{128\pi^6} & d=8 \end{cases}$$

- Monotonicity of the free energy is satisfied.

  (need to multiply $(-1)^{(d+1)/2}$ for odd d or $(-1)^{d/2}$ for even d)

Free energy on $\mathbb{H}^{p+1} \times \mathbb{S}^{q-1}$ and RG flow

- We can compute free energies on $\mathbb{H}^{p+1} \times \mathbb{S}^{q-1}$ although technically complicated.
- RG flow
  - $q = 2$: nontrivial b. c. $\Delta_{\mathsf{N1}} = (\Delta_+^{\ell \neq 1}, \Delta_-^{\ell=1})$, $\quad \Delta_{\mathsf{N2}} = (\Delta_+^{\ell \neq \pm 1}, \Delta_-^{\ell=\pm 1})$.
    RG flow from Δ_{N1} to Δ_+.
    $$F_{\Delta_+}[\mathbb{H}^{p+1} \times \mathbb{S}^1] - F_{\Delta_{\mathsf{N}1}}[\mathbb{H}^{p+1} \times \mathbb{S}^1] = F_{\ell=1,\mathsf{D}}[\mathbb{H}^{p+1} \times \mathbb{S}^1] - F_{\ell=1,\mathsf{N}}[\mathbb{H}^{p+1} \times \mathbb{S}^1] = -F[\mathbb{S}^p]$$
  - $q = 3, 4$: nontrivial b. c. $\Delta_{\mathsf{N}} = (\Delta_+^{\ell \neq 0}, \Delta_-^{\ell=0})$.
    RG flow from Δ_{N} to Δ_+,
    - $q = 3$: $F_{\Delta_+}[\mathbb{H}^{p+1} \times \mathbb{S}^2] - F_{\Delta_{\mathsf{N}}}[\mathbb{H}^{p+1} \times \mathbb{S}^2] = F_{\Delta_+}[\mathbb{H}^{p+1}] - F_{\Delta_-}[\mathbb{H}^{p+1}]$.
    - $q = 4$: $F_{\Delta_+}[\mathbb{H}^{p+1} \times \mathbb{S}^3] - F_{\Delta_{\mathsf{N}}}[\mathbb{H}^{p+1} \times \mathbb{S}^3] = -F[\mathbb{S}^p]$
- In all cases, the free energies decrease!!

Summary

- We consider a conformally coupled scalar field theory on $\mathbb{H}^{p+1} \times \mathbb{S}^{q-1}$.

  $\rightarrow$ With conformal boundary conditions, it can be regarded as defect CFT.

- We classified the allowed boundary condition on $\mathbb{H}^{p+1} \times \mathbb{S}^{q-1}$.

  For $q = 1, 2, 3, 4$, the nontrivial boundary condition is allowed.

- We consider RG flow triggered by a mass deformation on defect from the non-trivial b. c. (Neumann) to the trivial b. c. (Dirichlet), and we confirm a validity of our conjecture!

- Other direction

  - Free fermion [YS '21] (No non-trivial b. c. for free fields with spin ≥ 1)

  - Monodromy defect [Giombi et al. '21]

- Future direction

  Proof of defect C-theorem using quantum information quantity

Thank you for your attention!