Entanglement Entropy in Interacting Field Theories

Katsuta Sakai (KEK)

in collaboration with

Satoshi Iso (KEK & SOKENDAI),

Takato Mori (KEK & SOKENDAI)

・ PRD **103** 10, 105010 (2021)
・PRD **103** 12, 125019 (2021)
・ Symmetry **13** 7, 1221 (2021)
・ Work in progress

East Asian Joint Symposium on Fields and Strings 2021 (Osaka City U., Nov. 26, 2021)

1. Introduction

Entanglement Entropy (EE)

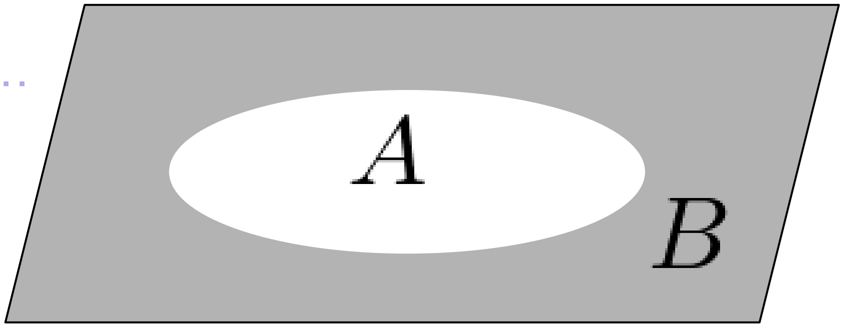
$$S_A = -\mathrm{Tr}_A[\rho_A \log(\rho_A)]$$

$$(\mathcal{H}_{\mathrm{tot}} = \mathcal{H}_A \otimes \mathcal{H}_B, \quad \rho_A = \mathrm{Tr}_B[\rho_{\mathrm{tot}}])$$

EE in field theory

･･･entanglement between spatially separated regions

- Quantum effects over the horizon (blackholes, de Sitter, …) [Solodukhin, '94], …
- Order parameter for quantum phase transition [Calabrese-Cardy '04], …
- Holographic counterpart of geometry [Ryu-Takayanagi, '06],…



1. Introduction

EE is well-studied in free field theories / CFTs.

What about the general interacting cases?

$\rightarrow$ Less understood

- Important to relate EE to realistic observables.

- Involved with the notion of radiative corrections

  Can we relate EE with renormalized quantities?

  cf) [Hertzberg, '10] at one loop

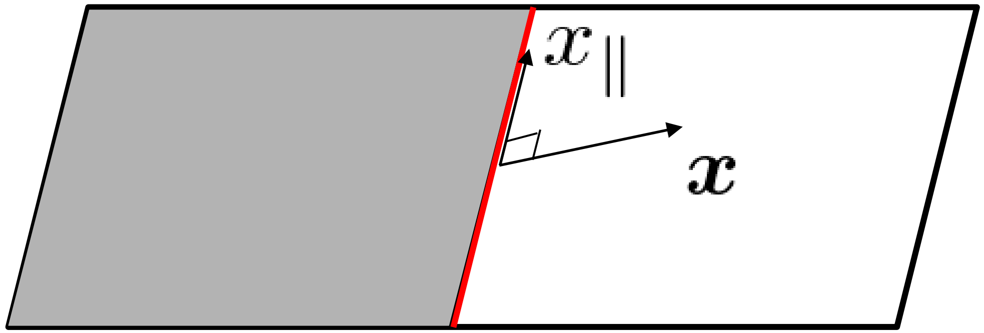
- No specific symmetry / simplification available.

  How can we evaluate it (analytically)?

1. Introduction

What we have done so far:

・To investigate EE in interacting (d+1)D field theory, in the case where the subregion is a flat half space.



・To extract a (dominant) part of EE which is expressed in terms of renormalized correlators of various operators.

$$S = \frac{V_{d-1}}{12} \int^{\epsilon^{-1}} \frac{d^{d-1} k_{||}}{(2\pi)^{d-1}} \mathrm{Tr} \log \left[\hat{G}(\boldsymbol{k} = \boldsymbol{0}, k_{||}) \right]$$

$$\hat{G}_{mn}(\boldsymbol{k} = \boldsymbol{0}, k_{||}) = \left\langle \, [\phi^m](\boldsymbol{0}, k_{||}) \;\; [\phi^n](\boldsymbol{0}, -k_{||}) \, \right\rangle$$

Plan of Talk

1. Introduction

2. Orbifold technique

3. Two specific contributions to EE

4. Discussion

5. Summary & Future work

2. Orbifold technique

A standard way to calculate EE ・・・ the replica method

$$S_A = \lim_{n\to 1} \frac{1}{1-n} \log \mathrm{Tr}[\rho_A^n] = -\frac{\partial}{\partial n} \mathrm{Tr}[\rho_A^n]\Big|_{n\to 1} ,$$

$$\mathrm{Tr}[\rho_A^n] = \frac{Z_n}{Z_1^n} \longleftarrow$$

: Partition function of the theory on Euclidean *n*-fold

Orbifold technique in free field cases

[Nishioka-Takayanagi, '06]

1. Replica trick

2. $n = 1/M$

⇨ theory on $\mathbb{R}^2/\mathbb{Z}_M \times \mathbb{R}^{d-1}$

free-energy

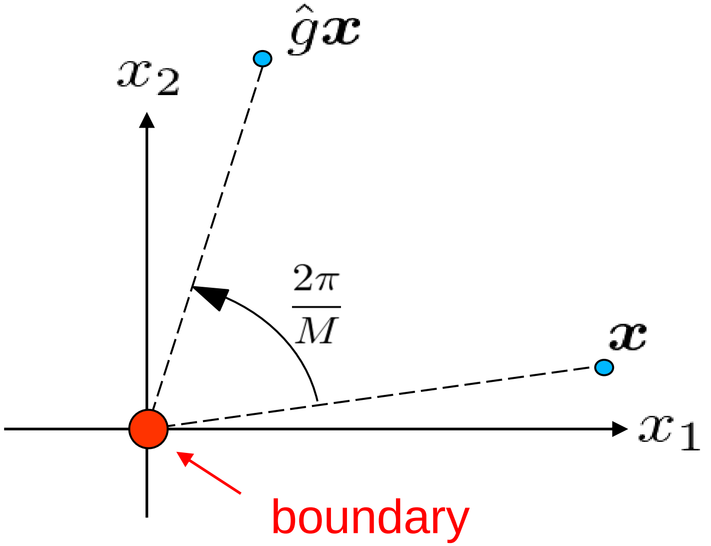
$$\boxed{S_A = -\frac{\partial (M F^{(M)})}{\partial M}\Big|_{M\to 1}}$$

2. Orbifold technique

Fields on $\mathbb{R}^2/\mathbb{Z}_M$ $\rightarrow$ those on $\mathbb{R}^2$ with a projection

$$\hat{P} = \frac{1}{M}\sum_{m=0}^{M-1} \hat{g}^m,$$

$$\hat{g}\left|\boldsymbol{x},\, x_\parallel\right\rangle = \left|\left(\frac{2\pi}{M}\text{-rot. of } \boldsymbol{x}\right),\, x_\parallel\right\rangle$$



$$F^{(M)} = \frac{1}{2}\mathrm{Tr}\left[\hat{P}\log(-\Box + m^2)\right]$$
$$= \frac{1}{2}\int \frac{d^2\boldsymbol{k}}{(2\pi)^2}\frac{d^{d-1}k_\parallel}{(2\pi)^{d-1}}\log(k^2+m^2)\,\underline{\left\langle \boldsymbol{k},\, k_\parallel\right|\hat{P}\left|\boldsymbol{k},\, k_\parallel\right\rangle}$$

easily calculable

$$\downarrow$$

$$S_A = -\frac{V_{d-1}}{12}\int^{1/\epsilon}\frac{d^{d-1}k_\parallel}{(2\pi)^{d-1}}\log\left[(k_\parallel^2+m^2)\epsilon^2\right] = \frac{V_{d-1}}{12}\int\frac{d^{d-1}k_\parallel}{(2\pi)^{d-1}}\log G_0(\boldsymbol{k}=\boldsymbol{0}, k_\parallel)$$

Interacting field theory on the orbifold

Ex): ϕ^4-theory

$$S = \int \frac{d^2\boldsymbol{x}}{M} d^{d-1}x_{\parallel} \left(\frac{1}{2}\phi \hat{P}(-\Box + m^2)\hat{P}\phi + \frac{\lambda}{4}(\hat{P}\phi)^4 \right)$$

$$\Downarrow$$

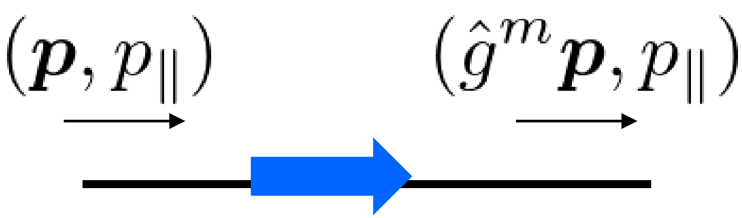
Propagator: $G_0^{(M)}(x,y) = M\,\hat{P}G_0(x,y)\hat{P}$

Flat space one

Vertex: $-\dfrac{\lambda}{4M}$

Interpretation of the propagator

$$
\begin{aligned}
G_0^{(M)}(x,y) &= \sum_{m=0}^{M-1} G_0(\hat{g}^m x - y) \\
&= \sum_{m=0}^{M-1} \int \frac{d^2\boldsymbol{p}}{(2\pi)^2} \frac{d^{d-1}p_\parallel}{(2\pi)^{d-1}} \frac{e^{ip_\parallel(x_\parallel - y_\parallel) + i\boldsymbol{p}(\hat{g}^m \boldsymbol{x} - \boldsymbol{y})}}{p^2+m^2} \\
&= \sum_{m=0}^{M-1} \int \frac{d^2\boldsymbol{p}}{(2\pi)^2} \frac{d^{d-1}p_\parallel}{(2\pi)^{d-1}} \frac{e^{ip_\parallel(x_\parallel - y_\parallel) + i((\hat{g}^m \boldsymbol{p})\boldsymbol{x} - \boldsymbol{p}\boldsymbol{y})}}{p^2+m^2}
\end{aligned}
$$



: Carrying a twisted momentum

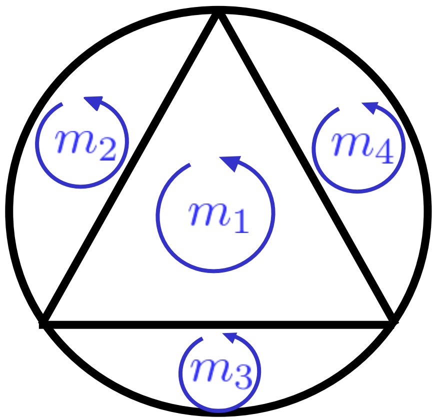
2. Orbifold technique

Free energy on the orbifold

$=$ Connected bubble diagrams with the projected loop momenta

Ex)

$$\sum_{m_1,\cdots,m_4}$$



$$= \frac{\mathrm{Area}(\partial A)}{M} \sum_{m_1,\cdots,m_4} \int \prod_{l=1}^{4} [d^2\boldsymbol{p}^{(l)} d^{d-1}p^{(l)}] I(p) \delta^2 \left(\sum_{l=1}^{4} (1-\hat{g}^{m_l}) \boldsymbol{p}^{(l)} \right)$$

$$= \frac{\mathrm{Area}(\partial A)}{M} \sum_{m_1,\cdots,m_4} \underline{\tilde{F}(m_1,\cdots,m_4)}$$

Free energy with "twisted" loop momenta

3. Two specific contributions to EE

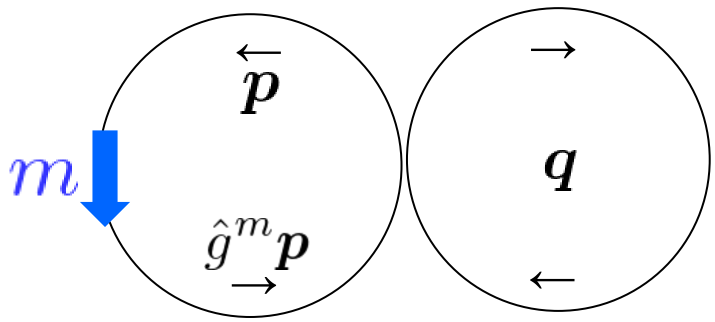
Calculation of $$\sum_{m_1,\cdots,m_4} \tilde{F}(m_1,\cdots,m_4)$$

- It is technically difficult.
- Still, we can extract some contributions of physical importance.

A single loop momentum twisted:

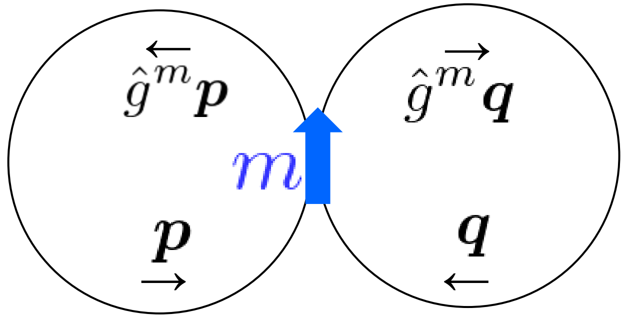
① on a propagator

→ Propagator contributions



② on a channel on a vertex

→ Vertex contributions

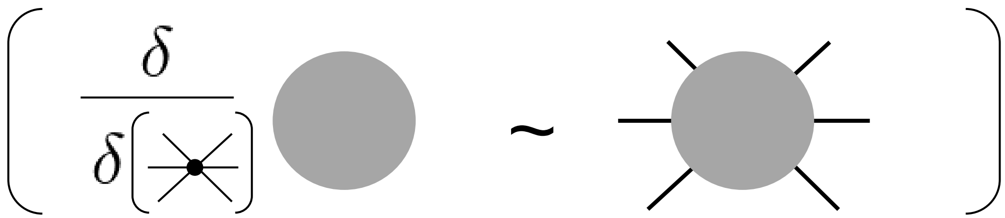


Both of them can be computated in the same way

3. Two specific contributions to EE

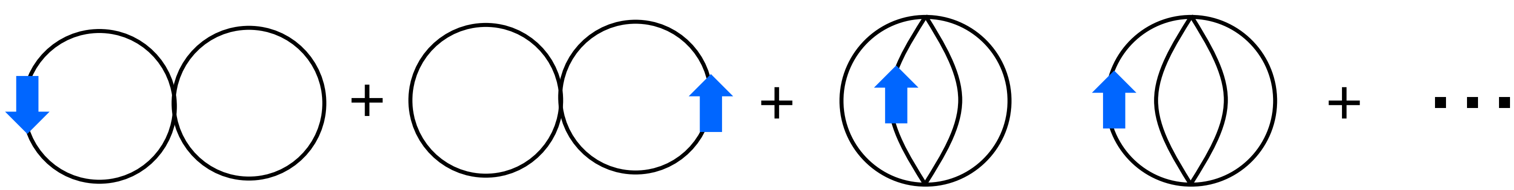
Using a general formula

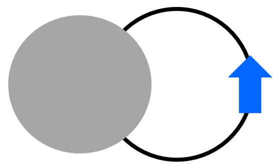
$$\frac{\delta F}{\delta G_0(x_1,\cdots,x_n)} \sim G(x_1,\cdots,x_n) \quad ,$$



we resolve each part on which the momentum is to be twisted,
and reconnect the Green function by the twisted part.

Ex) 2-pt. (= propagator contributions)



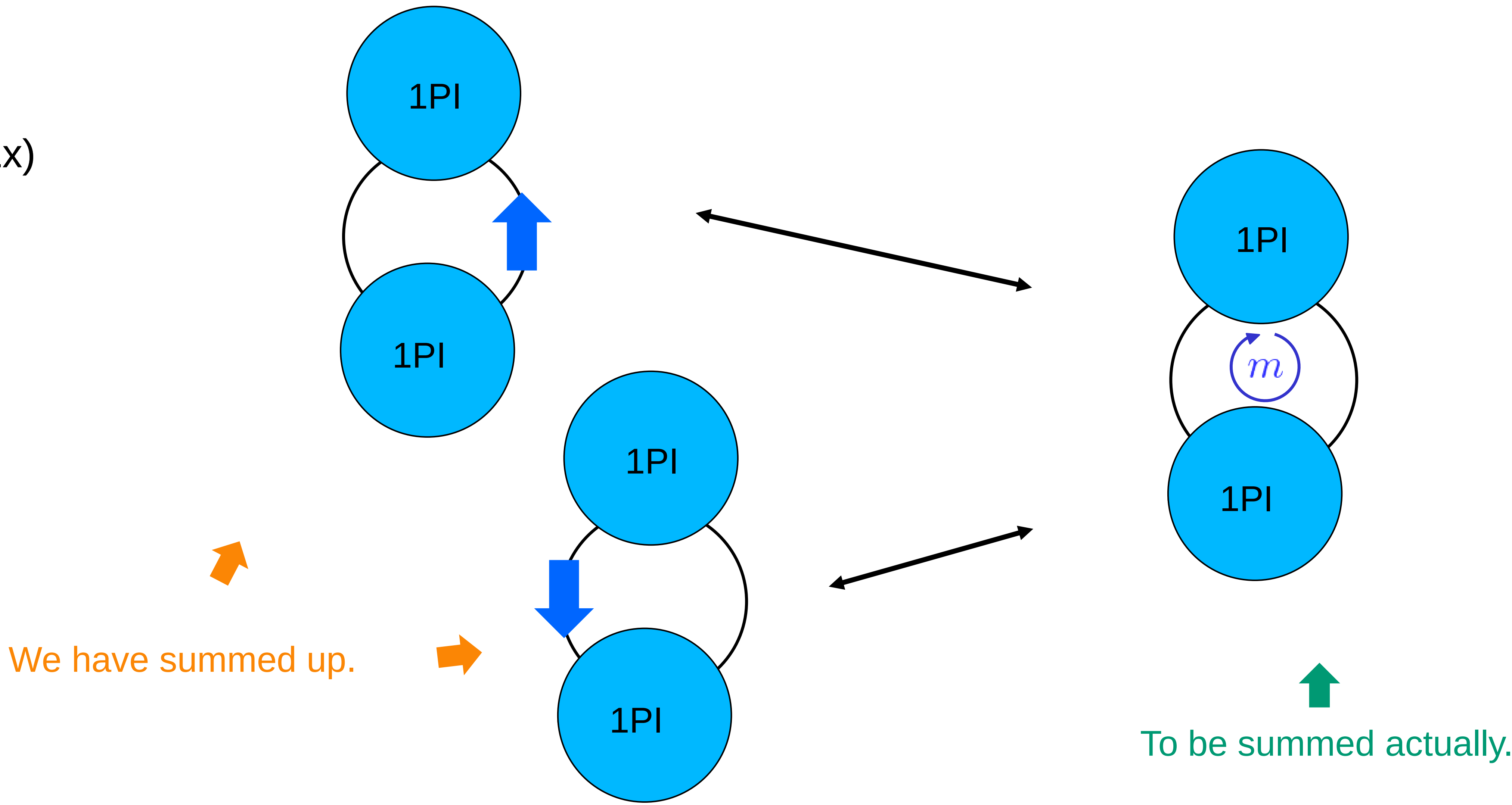
$\sim$ 

$$\rightarrow \frac{V_{d-1}}{24}\frac{M^2-1}{M}\int\frac{d^{d-1}k_{\parallel}}{(2\pi)^{d-1}}G_{\phi\phi}(\boldsymbol{k}=\boldsymbol{0},k_{\parallel})$$

3. Two specific contributions to EE

However, there’re redundant correspondences.

Ex)

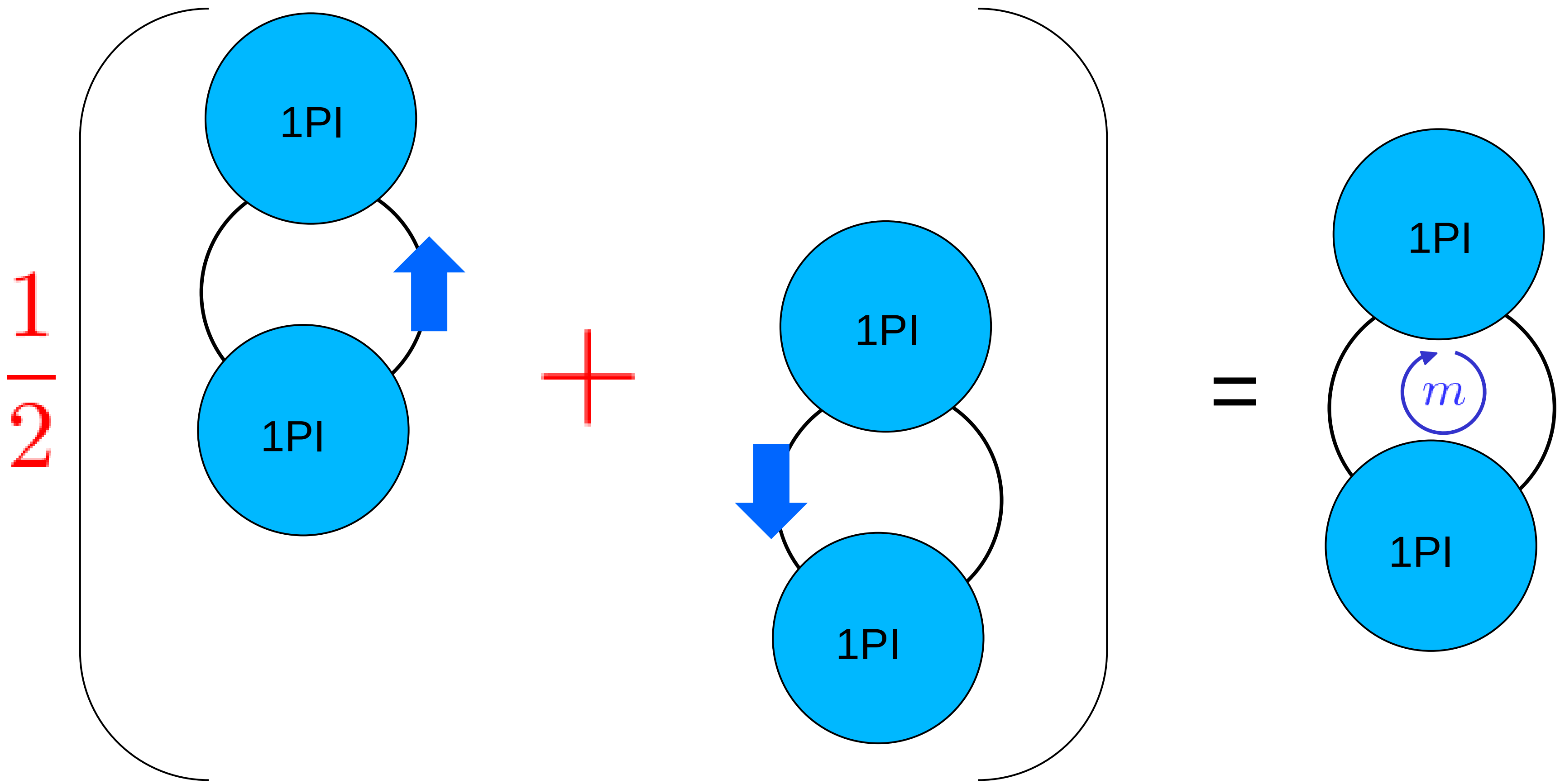


If we take both the twisted propagators, it will be doublecounting.

3. Two specific contributions to EE

The doublecounting is resolved by dividing.

Ex)



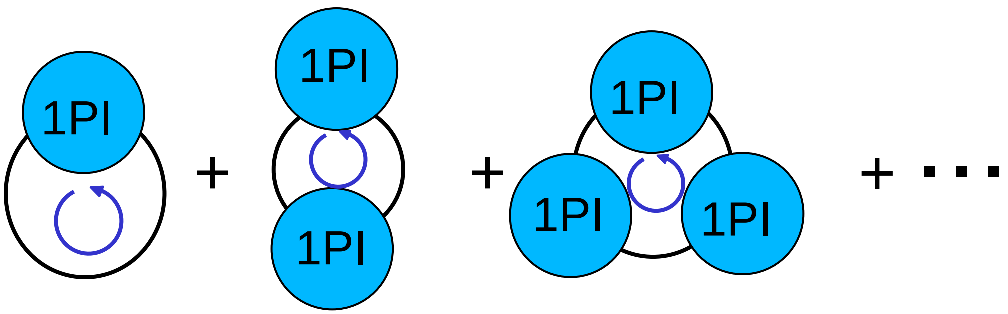
If we take both the twisted propagators, it will be doublecounting.

3. Two specific contributions to EE

free part
Including
overcounting

$$S_{2\mathrm{pt}}^{(\mathrm{naive})} = \frac{V_{d-1}}{12}\int \frac{d^{d-1}k_\parallel}{(2\pi)^{d-1}} G_{\phi\phi}(\boldsymbol{k}=\boldsymbol{0}, k_\parallel)$$

$$= \frac{V_{d-1}}{12}\int \frac{d^{d-1}k_\parallel}{(2\pi)^{d-1}} \left[1 + G_0\Sigma + (G_0\Sigma)^2 + (G_0\Sigma)^3 + \cdots\right] G_0$$

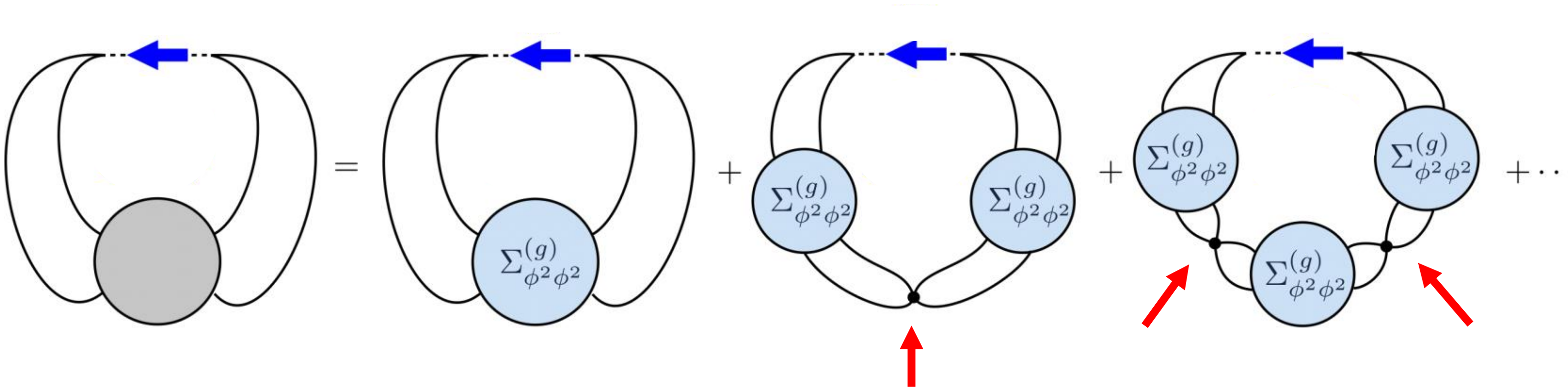


$$= \frac{V_{d-1}}{12}\int \frac{d^{d-1}k_\parallel}{(2\pi)^{d-1}} \left[1 + G_0\Sigma + \frac{1}{2}(G_0\Sigma)^2 + \frac{1}{3}(G_0\Sigma)^3 + \cdots + \log G_0\right]$$

$$= \frac{V_{d-1}}{12}\int \frac{d^{d-1}k_\parallel}{(2\pi)^{d-1}} \log\left[G_{\phi\phi}(\boldsymbol{k}=\boldsymbol{0}, k_\parallel)\right]$$

3. Two specific contributions to EE

4-pt in φ[4] theory



$$S^{(\mathrm{naive})}_{4\mathrm{pt}} = -\frac{V_{d-1}}{12}\int\frac{d^{d-1}k_\parallel}{(2\pi)^{d-1}}\frac{3\lambda_4}{2}\left[\Sigma^{(g)}_{\phi^2\phi^2} + \Sigma^{(g)}_{\phi^2\phi^2}\left(-\frac{3\lambda_4}{2}\right)\Sigma^{(g)}_{\phi^2\phi^2} + \Sigma^{(g)}_{\phi^2\phi^2}\left(-\frac{3\lambda_4}{2}\right)\Sigma^{(g)}_{\phi^2\phi^2}\left(-\frac{3\lambda_4}{2}\right)\Sigma^{(g)}_{\phi^2\phi^2} + \cdots\right]$$

$$S_{4\mathrm{pt}} = -\frac{V_{d-1}}{12}\int\frac{d^{d-1}k_\parallel}{(2\pi)^{d-1}}\frac{3\lambda_4}{2}\left[\Sigma^{(g)}_{\phi^2\phi^2} + \frac{1}{2}\Sigma^{(g)}_{\phi^2\phi^2}\left(-\frac{3\lambda_4}{2}\right)\Sigma^{(g)}_{\phi^2\phi^2} + \frac{1}{3}\Sigma^{(g)}_{\phi^2\phi^2}\left(-\frac{3\lambda_4}{2}\right)\Sigma^{(g)}_{\phi^2\phi^2}\left(-\frac{3\lambda_4}{2}\right)\Sigma^{(g)}_{\phi^2\phi^2} + \cdots\right]$$

$$= \frac{V_{d-1}}{12}\int\frac{d^{d-1}k_\parallel}{(2\pi)^{d-1}}\log G_{\phi^2\phi^2}(\boldsymbol{k}=\boldsymbol{0}, k_\parallel)$$

3. Two specific contributions to EE

We can generalize the analysis to the case where operators are mixed in vertices.

Ex) $\phi^6 \longrightarrow [\phi^2][\phi^4], \;\; [\phi^3][\phi^3]$

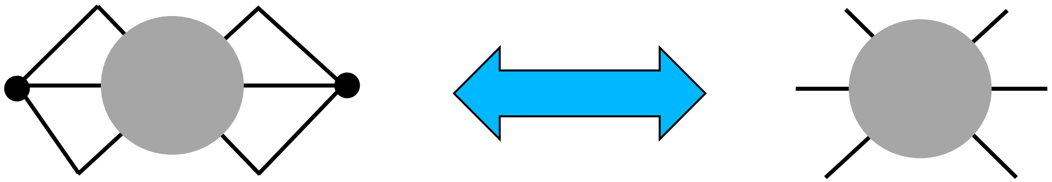
$$S = \frac{V_{d-1}}{12} \int^{\epsilon^{-1}} \frac{d^{d-1} k_{\|}}{(2\pi)^{d-1}} \mathrm{Tr} \log \left[\hat{G}(\boldsymbol{k} = \mathbf{0}, k_{\|}) \right]$$

$$\begin{aligned} \hat{G}_{mn}(\boldsymbol{k} = \mathbf{0}, k_{\|}) &= G_{\phi^m \phi^n}(\boldsymbol{k} = \mathbf{0}, k_{\|}) \\ &= \left\langle \, [\phi^m](\mathbf{0}, k_{\|}) \;\; [\phi^n](\mathbf{0}, -k_{\|}) \, \right\rangle \end{aligned}$$

4. Discussion

- What set operators appears?

  -- Those associated with connected Green function.



- What about the other contributions to EE?

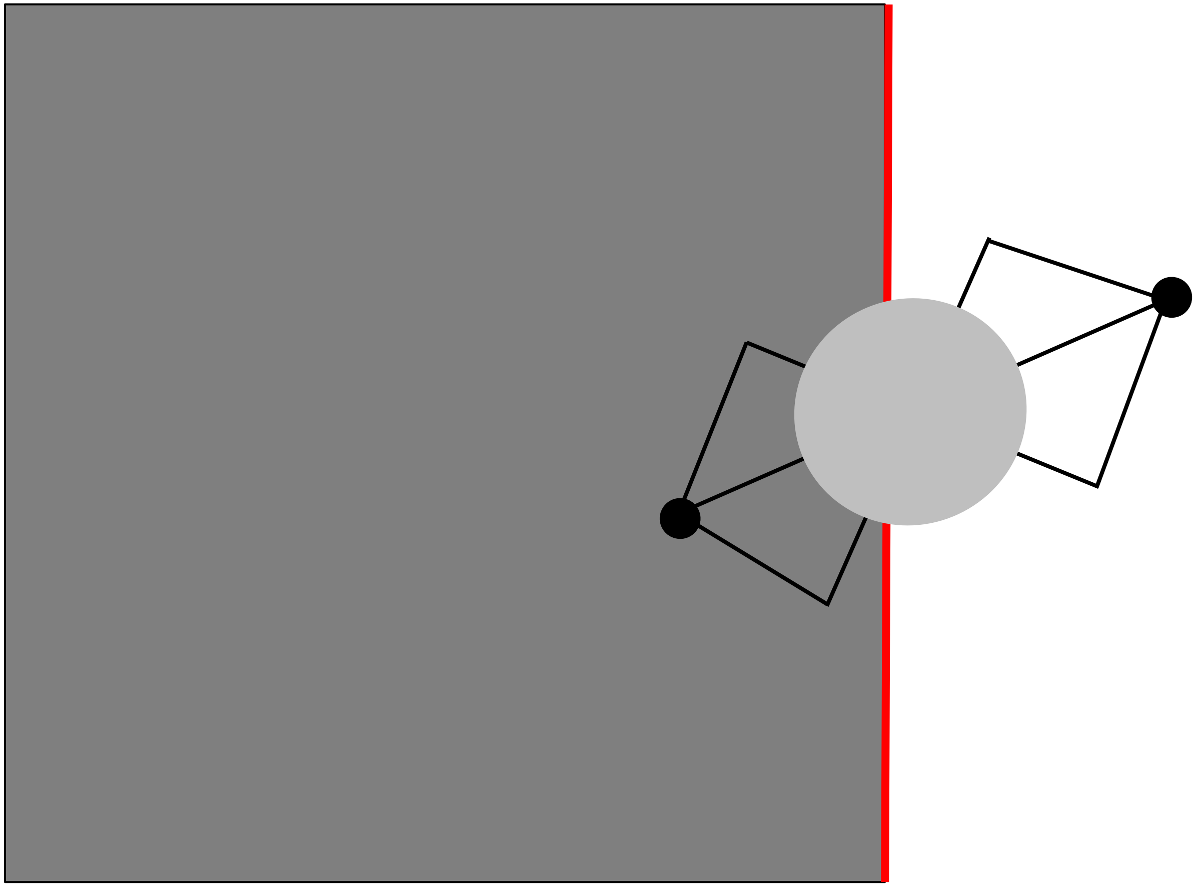
  -- They all appear as relatively higher-loop corrections.

  In particular, when we begin with an effective action, such contributions are expected to be negligible…

4. Discussion

- What is the interpret of the result?

  -- In the position space, it is a sum of correlation between subsystems via operators



5. Summary & Future work

Summary

- We have investigated EE in an interacting field theory with flat spatial boundary.

- Orbifold technique to interacting theories
  → EE includes a special contributions represented with renormalized two pt. functions of operators, and it would be dominant.
  → Measurable in a sense?

Future work

- Justification of the dominance
- More precise investigation about renormalization
- More generalization
  -- with spins and derivative couprings → OK.
  -- with general subregions → Ongoing work (we are formulating now)