Target space entanglement in quantum mechanics of fermions and matrices

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Outline of my results

- We consider quantum entanglement in the target spaces of quantum theories.
  - Entanglement entropy (EE) in target spaces can be defined by extending the conventional definition of EE.
  - I investigated the target space EE for quantum mechanics.

- I derived a formula of target space EE for QM of noninteracting fermions,

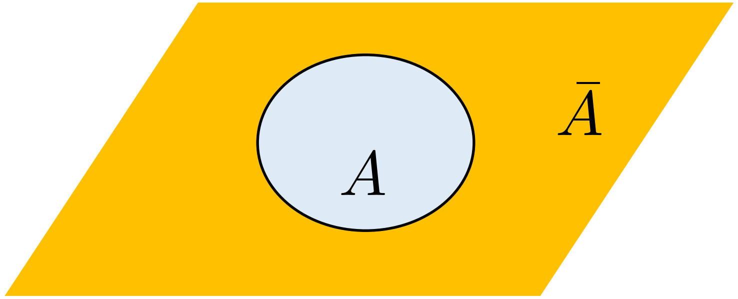
  $\approx$ one-matrix QM

  and analyzed a simple model.

Motivation

- Entanglement is a key concept in holography.
  - Ryu-Takayanagi

    base space entanglement in bdry ~ area in bulk



- Some holographic theories do not have the base space.

  e.g., BFSS model = M theory

  [Banks, Fischler, Shenker, Susskind (1996)]

  - matrix QM

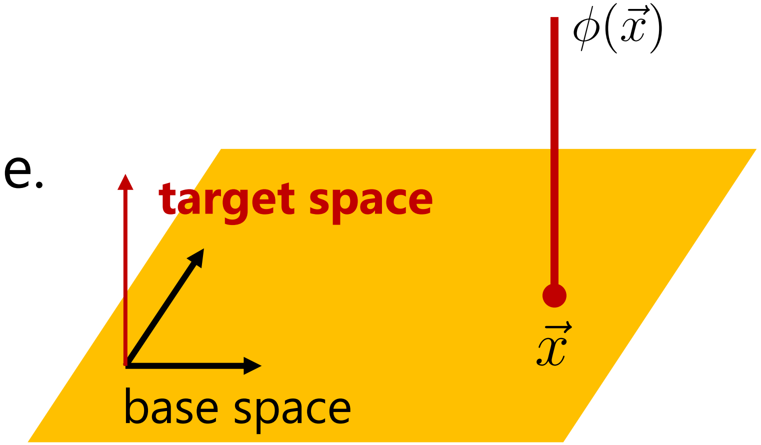
    (1+0)-dim QFT $X(t,\cancel{\vec{x}})$ $\Rightarrow$ no base space

➢ A new proposal: **Target space entanglement ~ Area in gravity**

[Das, Kaushal, Mandal, Trivedi (2020)], see also [Das, Kaushal, Liu, Mandal, Trivedi (2020)]

Target space entanglement

- For QFTs, we usually consider entanglement in the base space.



➡ **Target space entanglement** has not been considered so much.

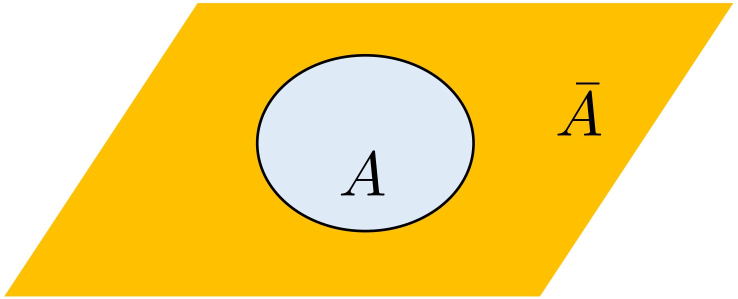
[Mazenc, Ranard (2019)], [Hampapura, Harper, Lawrence (2020)],
[Das, Kaushal, Mandal, Trivedi (2020)], [Das, Kaushal, Liu, Mandal, Trivedi (2020)] , [Frenkel, Hartnoll (2021)]

➢ Many basic properties have not yet been understood well.

Definition of entanglement entropy

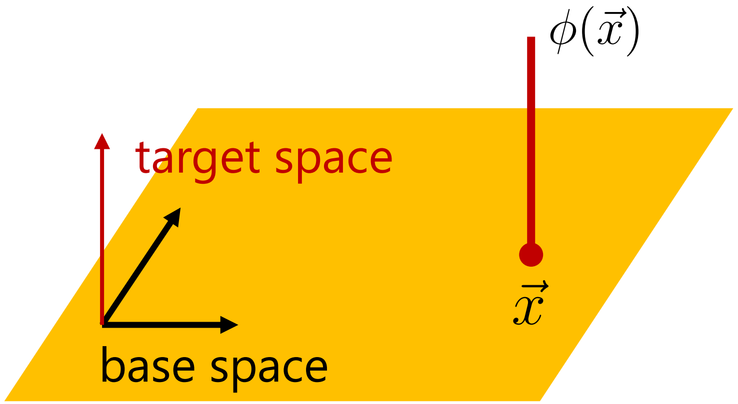
- conventional base space EE $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}, \quad (\mathcal{H}_A = \otimes_{x\in A}\mathcal{H}_x, \quad \mathcal{H}_{\bar{A}} = \otimes_{x\in\bar{A}}\mathcal{H}_x)$

density mat ρ ➡ $\rho_A = \mathrm{tr}_{\bar{A}}\,\rho$ ➡ $S_A = -\,\mathrm{tr}_A\,\rho_A \log \rho_A$



➢ **Problem**

Hilbert space is not tensor-factorized w.r.t. the target space. $\mathcal{H} \neq \otimes_{\phi\in\mathbb{R}}\mathcal{H}_\phi$



- Take another definition (an algebraic approach)
(used also in (lattice) gauge theories to define EE)

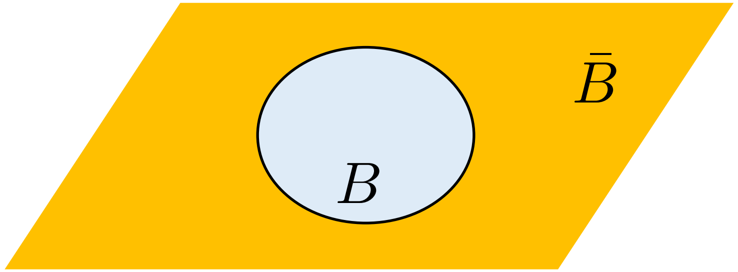
Entropy = Measure of uncertainty

- If we can only know **partial information** (i.e., use only a subset of operators), ``**entropy**'' is defined as a **measure of uncertainty** (or unknownness) about the whole info.

- (Geometrical) entanglement entropy

  Uncertainty for an observer who can probe only subregion

  $$\mathcal{L}(\mathcal{H}_B) \otimes 1_{\bar{B}} \subset \mathcal{L}(\mathcal{H})$$



- **Can define entropy for a subset of operators (subalgebra $\mathcal{A}$)**
  - given state (total density mat ρ)
  - restricted operators (take a subalgebra $\mathcal{A}$)

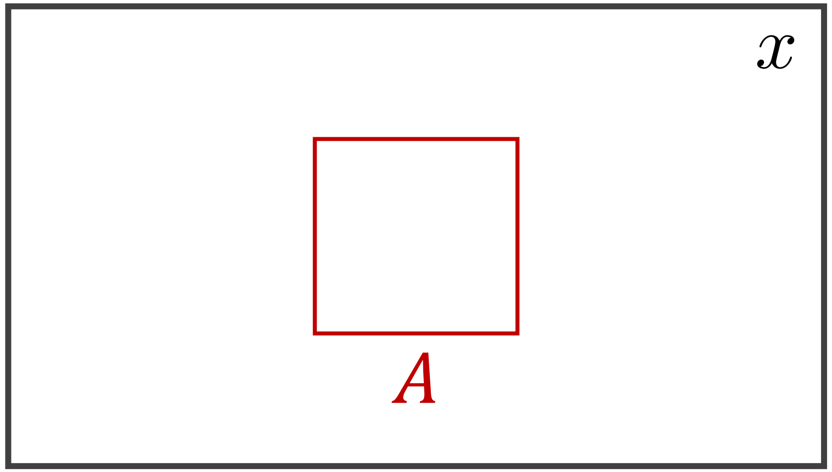
  $\Rightarrow$ $S_{\mathcal{A}}(\rho)$

  Today, I skip this part.

No need of tensor product structure.
This general def can be applied to target space EE.

QM of non-relativistic fermions

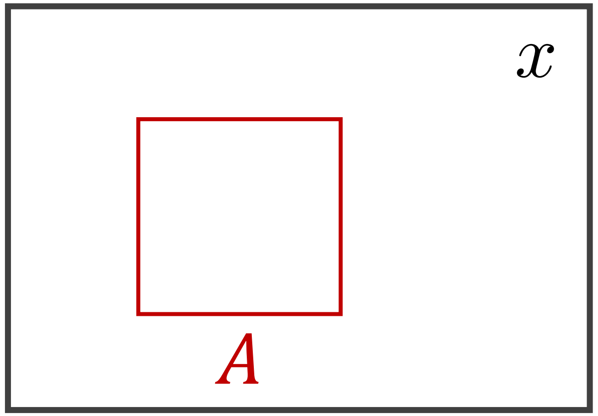
- Consider target space EE for QM of N fermions.
  (~ target space EE for $N\times N$ one-matrix QM)



Fermions with the Slater det wave functions

- Slater determinant $\psi(x_1, \ldots, x_N) = \frac{1}{\sqrt{N!}} \det\left(\chi_i(x_j)\right)$

$$\left[\int_{\mathrm{entire}} dx\, \chi_i(x)\chi_j^*(x) = \delta_{ij} \right]$$



Take subalgebra on A. Then, EE is as follows.

$$\Rightarrow \quad \boxed{S(A) = -\,\mathrm{tr}[X \log X + (1_N - X)\log(1_N - X)]}$$

- overlap matrix $$\boxed{X_{ij}(A) = \int_A dx\, \chi_i(x)\chi_j^*(x)}$$

Upper bound on EE of fermions

$$\boxed{S_A(\rho) \leq N \log 2}$$

- **Entropy is finite** if # of particle N is finite.

  Note: Dim of Hilbert space is ∞.

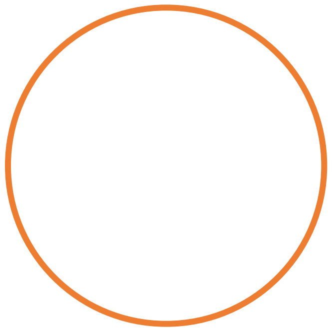
EE in QFTs is generally UV divergent.
QM is UV finite.

- the maximum entropy $N \log 2$ is **extensive (volume law)**

✓ This is too generic. EE for the ground state is smaller. **[sub-extensive (area law)]**

For ground state of 1-dim free fermi gas $S_A(\rho) \sim \dfrac{1}{3} \log N$ (see soon)

Fermi gas on circle

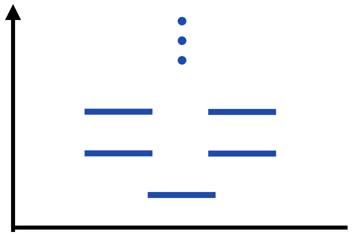


Length L

- 1-particle eigenfunction

$$\chi_i(x) = \frac{1}{\sqrt{L}} e^{\frac{2\pi i}{L} n_i x} \qquad (-L/2 \leq x \leq L/2)$$

$$n_1 = 0,\ n_2 = -1,\ n_3 = 1,\ n_4 = -2,\ n_5 = 2,\ \cdots$$



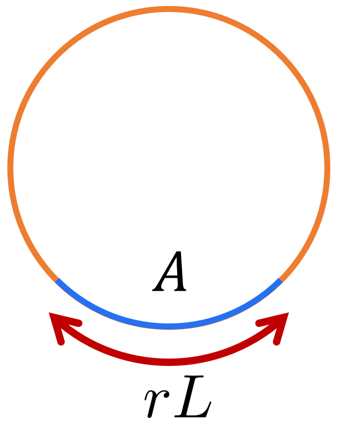
- ground state $\psi(x_1, \dots, x_N) = \frac{1}{\sqrt{N!}} \det\left(\chi_i(x_j)\right)$ (suppose N = odd)

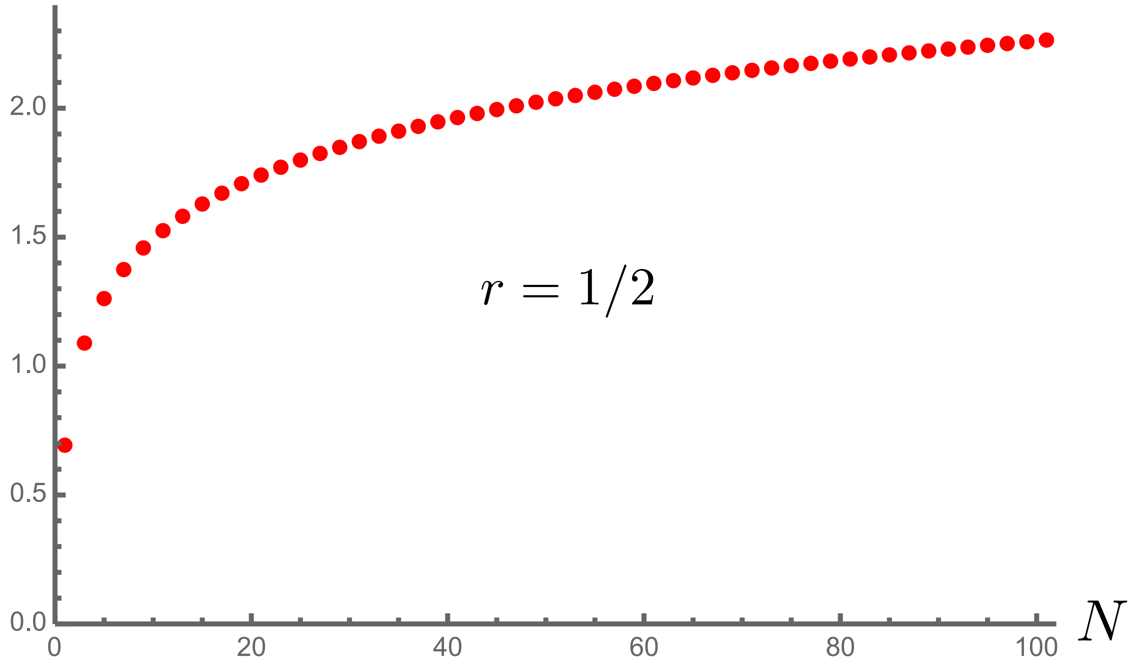
EE for a single interval

- EE of interval (length rL)

overlap matrix $X_{ij}(A) = \int_A dx\, \chi_i(x)\chi_j^*(x)$

$$\Rightarrow\quad S(A) = -\,\mathrm{tr}[X\log X + (1_N - X)\log(1_N - X)]$$



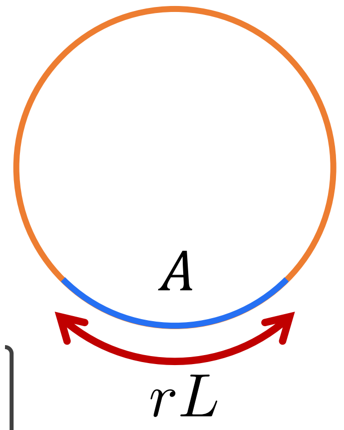


Not linear in N

Large N

- the asymptotic form at large N (same as the XX model)

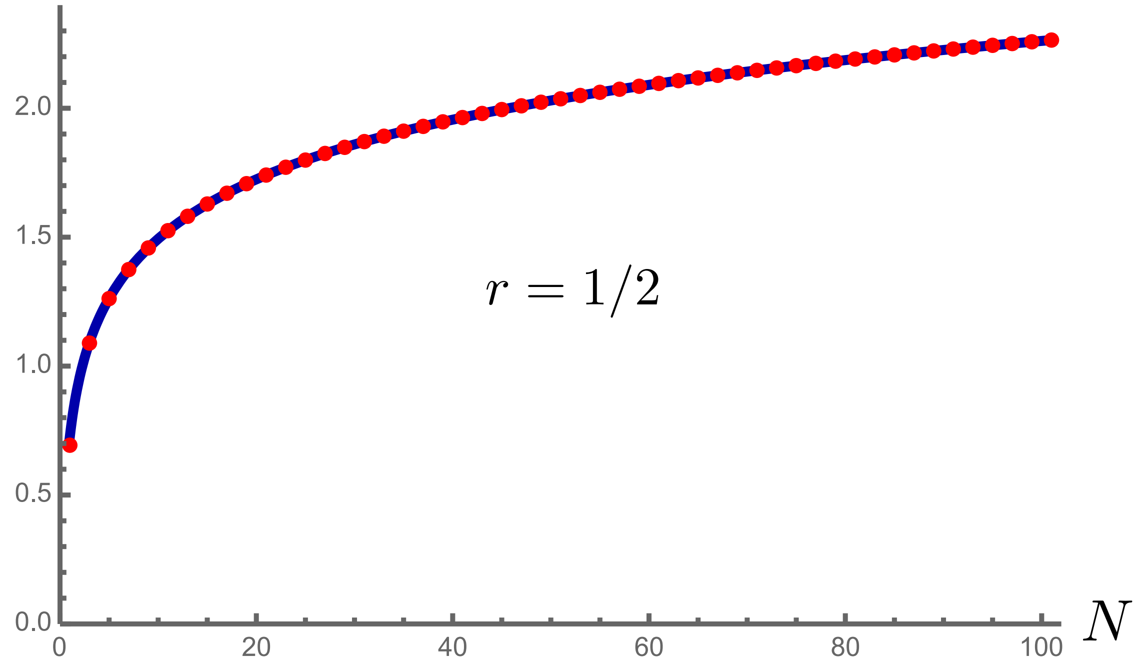
[Jin, Korepin (2004), Calabrese, Essler (2010)]



$$S \sim \frac{1}{3}\log[2N\sin(\pi r)] + \Upsilon_1$$

$$\left[\Upsilon_1 = i\int_{-\infty}^{\infty} dw \frac{\pi w}{\cosh^2(\pi w)} \log\frac{\Gamma\left(\frac{1}{2}+iw\right)}{\Gamma\left(\frac{1}{2}-iw\right)} \sim 0.495\right]$$

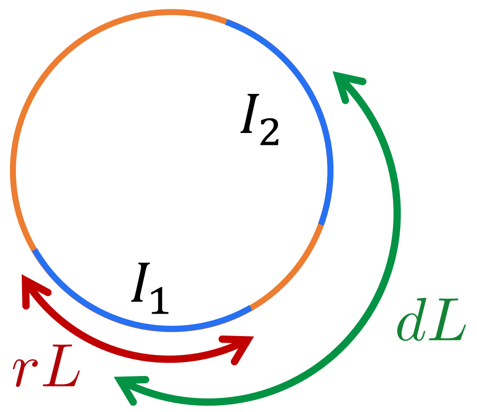
sub-extensive (area law)

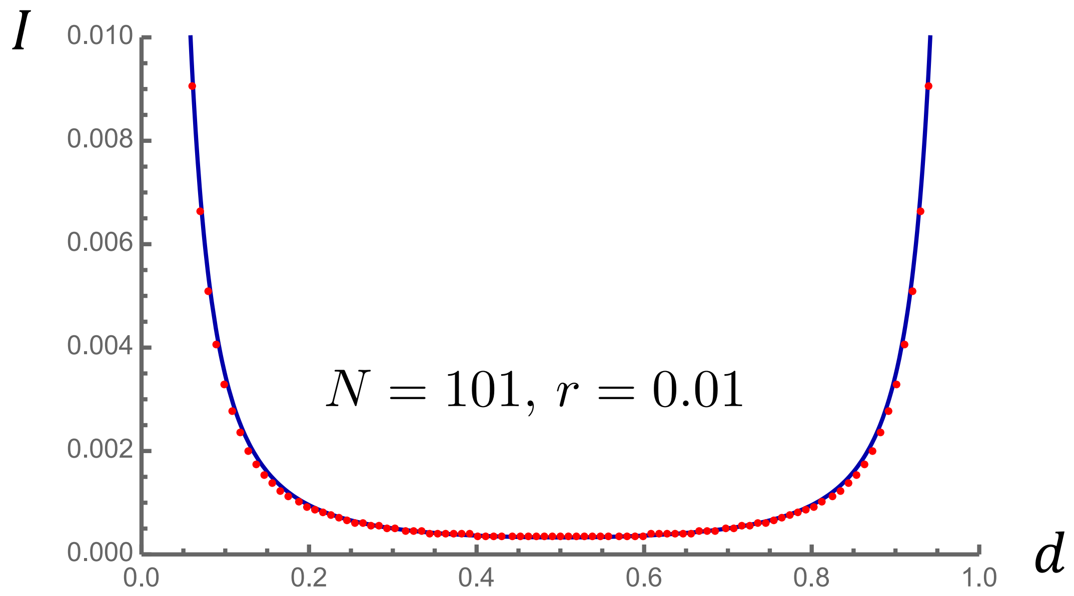


Mutual information

- **mutual information** of two intervals

$$I(I_1;I_2)=S(I_1)+S(I_2)-S(I_1\cup I_2)$$ （**correlation of two regions**）





analytic result （large N）

$$I(I_1;I_2)\sim\frac{1}{3}\log\frac{\sin^2(\pi d)}{\sin[\pi(d+r)]\sin[\pi(d-r)]}$$

finite at $N\to\infty$.

UV finite also in QFTs.

Agrees with the result of a CFT [Calabrese, Cardy (2004)]
(free compact boson at self-dual radius).
Why?

Summary

- Generalized def of entanglement based on the algebraic approach
  - We can define EE associated with a subalgebra $\Rightarrow$ Applicable to various situations

- Target space EE for identical particles

- 1d free fermi gas
  - numerical and analytical (large N) results

➢ want to consider multi-matrix QM

➢ Implications for holography?